ON THE DEGREE OF CONVERGENCE OF LAPLACE'S SERIES*

BY

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§ 1. Introduction.

Let S denote a sphere of unit radius, and refer its points to a geographical system of coördinates, θ being the polar distance and φ the longitude; the north pole and the zero meridian may be fixed arbitrarily. Further let $f(\theta, \varphi)$ be a uniform function of the position on the sphere, and suppose this function to be absolutely integrable in the Riemann sense, so that

$$\int_S |f(\theta, \varphi)|\, d\sigma$$

exists, where $d\sigma = \sin\theta\, d\theta\, d\varphi$ is the surface element of S. We finally denote by θ', φ' any point on S, the corresponding surface element being $d\sigma'$, and by γ the angle between the two vectors from the center of the sphere to the points θ, φ and θ', φ' respectively (or, in other words, the distance between the two points in question, measured on the great circle joining them). This angle γ is completely determined by the condition $0 \leqq \gamma \leqq \pi$, and we have

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi' - \varphi).$$

Then *the Laplace series corresponding to* $f(\theta, \varphi)$ is

$$(1) \qquad f(\theta, \varphi) \sim \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_S f(\theta', \varphi')\, P_n(\cos\gamma)\, d\sigma',$$

P_n being the nth Legendre polynomial; the equivalence sign $\sim$ signifies that the series is considered in a purely formal way, and implies no statement in regard to its convergence.

An important particular case is obtained by supposing $f(\theta, \varphi)$ independent of φ; writing $\cos\theta = x$ and $f(\theta, \varphi) = f(\cos\theta) = f(x)$, the Laplace series (1) becomes *the Legendre series corresponding to* $f(x)$:

$$(2) \qquad f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x), \qquad c_n = \frac{2n+1}{2} \int_{-1}^{+1} f(y)\, P_n(y)\, dy.$$

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The present paper is concerned with the degree of approximation obtained by replacing $f(\theta, \varphi)$ by the sum of the first $n+1$ terms in its Laplace series, or in other words, with the investigation of the function

$$(3) \quad r_n\{f(\theta, \varphi)\} = f(\theta, \varphi) - \sum_{\nu=0}^{n} \frac{2\nu+1}{4\pi} \int_S f(\theta', \varphi') P_\nu(\cos\gamma)\, d\sigma'$$

under the assumption that for any two points θ, φ and θ', φ' the function $f(\theta, \varphi)$ satisfies a generalized Lipschitz condition of the form

$$(4) \quad |f(\theta', \varphi') - f(\theta, \varphi)| \leqq \omega(\gamma),$$

where, for γ increasing, $\omega(\gamma)$ never decreases, and $\omega(\gamma)/\gamma$ never increases, while $\lim_{\gamma=0} \omega(\gamma) = 0$. On account of the last condition, $f(\theta, \varphi)$ is obviously continuous all over the sphere S.

In a recent paper, Jackson* has treated the corresponding problem for the Legendre series (2), assuming that $f(x)$ satisfies the condition

$$(5) \quad |f(x_2) - f(x_1)| \leqq \omega(|x_2 - x_1|), \quad -1 \leqq x_1 \leqq 1, \quad -1 \leqq x_2 \leqq 1,$$

the function ω being subject to the same restrictions as above. He confines his investigation, however, to the points of the interval

$$(6) \quad -1 + \epsilon \leqq x \leqq 1 - \epsilon,$$

where ϵ is any *positive* quantity. The methods of the present paper apply, on convenient specialization, to Legendre's series for $x = \pm 1$; whenever the occasion arises, we shall compare our results with those of Jackson and thus bring out the fundamental difference between the behavior of the Legendre series in the interior points of the interval $(-1, 1)$ and in the end points.

Section 2 of the present paper gives a lemma of fundamental importance: supposing $f(\theta, \varphi)$ in (1) limited by the condition that $|f(\theta, \varphi)| \leqq 1$ at every point of S, then there exists a function $f(\theta, \varphi)$ of this kind which makes the absolute value of the sum of the first $n+1$ terms in Laplace's series a maximum ρ_n at a given point θ, φ, and it is shown that these constants ρ_n, which we shall call the Lebesgue constants for Laplace's series,† tend towards infinity with n as $\sqrt{n}$,‡ or more accurately, that §

* Dunham Jackson, I. *On the Degree of Convergence of the Development of a Continuous Function According to Legendre's Polynomials*, these Transactions, vol. 13 (1912), pp. 305–318.

† Lebesgue first showed the fundamental importance of the corresponding constants for Fourier's series; see *Leçons sur les séries trigonométriques*, Paris, 1906, § 45.

‡ Jackson shows (l. c., pp. 306–309) that the corresponding maximum for Legendre's series in the interval (6) is inferior to a constant multiple of $\log n$.

§ The first proof of this relation may be found in § 2 of my paper *Über die Laplace'sche Reihe* (Mathematische Annalen, vol. 74 (1913), pp. 213–270); the present proof contains several simplifications and also some new developments which will be used in § 4.

$$\lim_{n=\infty} \frac{\rho_n}{\sqrt{n}} = 2\sqrt{\frac{2}{\pi}}.$$

In § 3 it will be shown that $f(\theta, \varphi)$ may be approximated by a sum $T_n(\theta, \varphi)$ of spherical harmonics of degree not exceeding n in such a way that

$$|f(\theta, \varphi) - T_n(\theta, \varphi)| \leqq K\omega\left(\frac{1}{n}\right) \quad (n = 1, 2, 3, \cdots)$$

at every point of the sphere S, K being a constant independent of n. The method of proof is closely related to that of Jackson* for the corresponding problem regarding $f(x)$.

In § 4, the preceding results are used to show that for all points of S

$$|r_n\{f(\theta, \varphi)\}| \leqq K'\omega\left(\frac{1}{n}\right)\sqrt{n} \quad (n = 1, 2, 3, \cdots),$$

K' being a constant independent of n, and that, conversely, if $\Omega(\gamma)$ is a function subject to the same restrictions as $\omega(\gamma)$ and furthermore

$$\lim_{\gamma=0} \frac{\Omega(\gamma)}{\omega(\gamma)} = 0,$$

then there exists a function $f(\theta, \varphi)$ satisfying (4) and such that at a given point θ_0, φ_0 the inequality

$$|r_n\{f(\theta_0, \varphi_0)\}| \leqq K''\Omega\left(\frac{1}{n}\right)\sqrt{n}$$

is satisfied for an infinite number of values of n, the constant K'' being independent of n.†

§ 2. The Lebesgue constants for Laplace's series.

The sum of the first $n+1$ terms in (1) is equal to

$$s_n\{f(\theta, \varphi)\} = \frac{1}{4\pi}\int_S f(\theta', \varphi')\, s_n(\cos\gamma)\, d\omega'$$

where

$$s_n(x) = \sum_{\nu=0}^{n} (2\nu + 1) P_\nu(x). \tag{7}$$

Limiting $f(\theta, \varphi)$ to the class of absolutely integrable functions which are not

* Dunham Jackson, II. *Uber die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung.* Inaugural-Dissertation, Göttingen, 1911. III. *On Approximation by Trigonometric Sums and Polynomials*, these Transactions, vol. 13 (1912), pp. 491–515.

† For $f(x)$ and x limited by (6), $\sqrt{n}$ may be replaced by $\log n$ and $\Omega(1/n)$ by $\omega(1/n)$ in the two statements regarding r_n. (Jackson, I, theorems 1 and 3.)

greater than unity in absolute value at every point of S, we see at once that

$$|s_n\{f(\theta,\varphi)\}| \leqq \frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\,d\sigma',$$

and the absolute maximum

$$\rho_n(\theta,\varphi) = \text{max.}\,|s_n\{f(\theta,\varphi)\}| = \frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\,d\sigma'$$

is reached at the point θ, φ by defining f so that

$$f(\theta',\varphi') = \text{sgn.}\, s_n(\cos\gamma),$$

where, in the notation of Kronecker,

$$\text{sgn.}\, a = \begin{cases} +1, & a>0 \\ 0, & a=0 \\ -1, & a<0. \end{cases}$$

Moving the north pole of our coördinate system to the point θ, φ, we obtain $\gamma = \theta'$ and

$$\rho_n(\theta,\varphi) = \frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\,d\sigma' = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} |s_n(\cos\theta')|\sin\theta'\,d\theta'\,d\varphi'$$
$$= \tfrac{1}{2}\int_0^{\pi} |s_n(\cos\theta')|\sin\theta'\,d\theta',$$

so that $\rho_n(\theta,\varphi) = \rho_n$ is independent of θ, φ; writing $\cos\theta' = x$, we find

$$\rho_n = \tfrac{1}{2}\int_{-1}^{+1} |s_n(x)|\,dx, \qquad (n = 0, 1, 2, \cdots). \tag{8}$$

In order to investigate the order of magnitude of ρ_n with respect to n, we begin by developing an asymptotic expression for $s_n(\cos\theta)$, which will furnish approximations to the roots of the equation $s_n(\cos\theta) = 0$.

From the well-known equation of definition of Legendre's polynomials

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x)\,z^n, \qquad |z|<1, \tag{9}$$

where the square root equals unity for $z = 0$, the equally well-known formula

$$\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)\,P_n(x)\,z^n$$

is easily obtained, and on multiplying both sides by

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

and using (7), it is seen at once that

$$\frac{1+z}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} s_n(x)\, z^n, \qquad |z|<1. \tag{10}$$

If we write $1/z$ instead of z and introduce $x = \cos\theta$, this becomes

$$\frac{z^2(z+1)}{(z-e^{\theta i})^{3/2}(z-e^{-\theta i})^{3/2}} = \sum_{n=0}^{\infty} s_n(\cos\theta)\, z^{-n}, \qquad |z|>1, \tag{11}$$

the radicals being determined so that their real parts tend toward $+\infty$ when z tends toward infinity by real and positive values. Then the application of Cauchy's theorem gives

$$s_n(\cos\theta) = \frac{1}{2\pi i}\int_C \frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz, \tag{11}$$

the contour of integration C enclosing both of the singular points $z = e^{\theta i}$ and $z = e^{-\theta i}$. Now let ϵ be a positive quantity which will later be made to tend toward zero, and let C consist of the circuit C_1 around $z = e^{\theta i}$, composed of the three parts: (1) the straight line from $z = 0$ to $z = (1-\epsilon)e^{\theta i}$, (2) the circle $z = e^{\theta i} + \epsilon e^{\phi i}$, $\theta - \pi \leqq \phi \leqq \theta + \pi$, (3) the straight line from $z = (1-\epsilon)e^{\theta i}$ to $z = 0$, followed by the circuit C_2 around $z = e^{-\theta i}$, composed of the three parts: (4) the straight line from $z = 0$ to $z = (1-\epsilon)e^{-\theta i}$, (5) the circle $z = e^{-\theta i} + \epsilon e^{\phi i}$, $-\theta - \pi \leqq \phi \leqq \pi - \theta$, (6) the straight line from $z = (1-\epsilon)e^{-\theta i}$ to $z = 0$.

To make the passage to the limit $\epsilon = 0$ possible, (11) must be transformed so that the integrand in (11) becomes infinite of an order less than one at the singular points, and to this purpose we use the identity

$$\frac{z+1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}} = -\frac{1}{z-1}\left[\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}} + 2z\frac{d}{dz}\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\right]$$

which gives, upon integration by parts,

$$\begin{aligned}\int \frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz &= -\int \frac{z^{n+1}}{(z-1)(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz \\ &\quad - 2\int \frac{z^{n+2}}{(z-1)}\frac{d}{dz}\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz \\ &= -\frac{2z^{n+2}}{z-1}\cdot\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}} \\ &\quad + \int \frac{(2n+3)z^{n+1}}{(z-1)(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz \\ &\quad - \int \frac{2z^{n+2}}{(z-1)^2(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz.\end{aligned}$$

Integrate over the circuit C_1; the expression outside of the integral signs vanishes at the beginning and the end of C_1 $(z = 0)$; making $\epsilon = 0$, the integrals over (2) vanish, and those over (3) become equal to the corresponding ones over (1), owing to the fact that $(z - e^{\theta i})^{\frac{1}{2}}$ changes its sign when z describes the circle (2). We therefore obtain

$$\frac{1}{2\pi i}\int_{C_1}\frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz = \frac{1}{\pi i}\int_0^{e^{\theta i}}\frac{(2n+3)z^{n+1}}{(z-1)(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz$$

$$-\frac{1}{\pi i}\int_0^{e^{\theta i}}\frac{2z^{n+2}}{(z-1)^2(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz,$$

the path of integration being the straight line from $z = 0$ to $z = e^{\theta i}$, and the initial values of the radicals for $z = 0$ having their real parts positive (being positive for $z = +\infty$ and never passing through zero as z moves along the real axis). We now introduce a new variable of integration, making

$$z = e^{\theta i}(1-u), \qquad 0 \leqq u \leqq 1;$$

then, on account of the manner of fixing the values of the radicals for $z = 0$,

$$(z-e^{\theta i})^{\frac{1}{2}} = e^{\frac{\theta-\pi}{2}i}\cdot u^{\frac{1}{2}}, \qquad u^{\frac{1}{2}} \geqq 0,$$

$$(z-e^{-\theta i})^{\frac{1}{2}} = \sqrt{2\sin\theta}\cdot e^{\frac{\pi}{4}i}\left(1-\frac{ue^{\left(\theta-\frac{\pi}{2}\right)i}}{2\sin\theta}\right)^{\frac{1}{2}},$$

where

$$\left(1-\frac{ue^{\left(\theta-\frac{\pi}{2}\right)i}}{2\sin\theta}\right)^{\frac{1}{2}} = +1 \quad \text{for} \quad u = 0.$$

Introducing all this in the integrals above, and writing

$$\alpha = \frac{e^{\frac{\theta-\pi}{2}}}{2\sin\frac{\theta}{2}}, \qquad \beta = \frac{e^{\left(\theta-\frac{\pi}{2}\right)i}}{2\sin\theta}, \tag{12}$$

we obtain

$$\frac{1}{2\pi i}\int_{C_1}\frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz = \frac{2n+3}{\pi}\cdot\frac{e^{(n+1)\theta i-\frac{1}{2}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}\int_0^1\frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}}\,du$$

$$-\frac{2}{\pi}\cdot\frac{e^{(n+\frac{1}{2})\theta i-\frac{1}{2}\pi i}}{\left(2\sin\frac{\theta}{2}\right)^2(2\sin\theta)^{\frac{1}{2}}}\int_0^1\frac{u^{-\frac{1}{2}}(1-u)^{n+2}}{(1-\alpha u)^2(1-\beta u)^{\frac{1}{2}}}\,du.$$

In exactly the same way it is seen that the corresponding integral over the

circuit C_2 has a value which is conjugate to the one just obtained for C_1, so that, by (11),

$$s_n(\cos\theta) = \frac{2(2n+3)}{\pi} R \frac{e^{(n+1)\theta i - \frac{1}{2}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} du$$

(13)

$$- \frac{4}{\pi} R \frac{e^{(n+\frac{1}{2})\theta i - \frac{1}{2}\pi i}}{\left(2\sin\frac{\theta}{2}\right)^2 (2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+2}}{(1-\alpha u)^2(1-\beta u)^{\frac{1}{2}}} du,$$

where R denotes the real part of the expression following it.

This formula will now easily give the desired asymptotic expression for $s_n(\cos\theta)$. On account of (12), we have for $0 \leqq u \leqq 1$

$$|1-\alpha u|^2 = \left(1-\frac{u}{2}\right)^2 + \frac{u^2}{4}\cot^2\frac{\theta}{2} \geqq \left(1-\frac{u}{2}\right)^2 \geqq \frac{1}{4},$$

$$|1-\beta u|^2 = \left(1-\frac{u}{2}\right)^2 + \frac{u^2}{4}\cot^2\theta \geqq \left(1-\frac{u}{2}\right)^2 \geqq \frac{1}{4},$$

so that

$$\frac{1}{|1-\alpha u|} \leqq 2, \qquad \frac{1}{|1-\beta u|} \leqq 2; \tag{14}$$

furthermore the identity

$$\frac{1}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} - 1 = \int_0^u \left(\frac{\alpha}{(1-\alpha u)^2(1-\beta u)^{\frac{1}{2}}} + \frac{1}{2}\cdot\frac{\beta}{(1-\alpha u)(1-\beta u)^{\frac{3}{2}}} \right) du$$

shows, by the aid of (14), that

$$\left| \frac{1}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} - 1 \right| < \int_0^u (2^{\frac{3}{2}}|\alpha| + 2^{\frac{3}{2}}|\beta|) du = \left(\frac{2^{\frac{3}{2}}}{2\sin\frac{\theta}{2}} + \frac{2^{\frac{3}{2}}}{2\sin\theta} \right) u < \frac{8u}{\sin\theta}.$$

From the first integral in (13) we now obtain

$$\left| \frac{2(2n+3)}{\pi} R \frac{e^{(n+1)\theta i - \frac{1}{2}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} du \right.$$

$$\left. - \frac{2(2n+3)}{\pi} R \frac{e^{(n+1)\theta i - \frac{1}{2}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \int_0^1 u^{-\frac{1}{2}}(1-u)^{n+1} du \right|$$

$$< \frac{2(2n+3)}{\pi} \cdot \frac{1}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \cdot \frac{8}{\sin\theta} \int_0^1 u^{\frac{1}{2}}(1-u)^{n+1} du,$$

or evaluating the Euler integrals of the first kind occurring in this formula and

reducing the result by means of $\Gamma(x+1) = x\,\Gamma(x)$,

$$\left|\frac{2(2n+3)}{\pi} R \frac{e^{(n+1)\theta i - \frac{1}{2}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}}\,du \right.$$

$$\left. - \frac{4}{\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)} \cdot \frac{\sin\left((n+1)\theta - \frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}\right|$$

$$< \frac{4}{\pi}\left(n+\frac{3}{2}\right)\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{7}{2}\right)} \cdot \frac{16}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}$$

$$< \frac{4}{\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)} \cdot \frac{8}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}.$$

The second integral in (13) may be estimated directly by (14), which gives

$$\left|\frac{4}{\pi} R \frac{e^{(n+\frac{1}{2})\theta i - \pi i}}{\left(2\sin\frac{\theta}{2}\right)^2 (2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-1}(1-u)^{n+2}}{(1-\alpha u)^2(1-\beta u)^{\frac{1}{2}}}\,du\right|$$

$$< \frac{4}{\pi} \cdot \frac{2^2 \cdot 2^{\frac{1}{2}}}{\left(2\sin\frac{\theta}{2}\right)^2 (2\sin\theta)^{\frac{1}{2}}} \int_0^1 u^{-\frac{1}{2}}(i-u)^{n+2}\,du$$

$$= \frac{4}{\pi} \cdot \frac{2^{\frac{1}{2}}}{\left(2\sin\frac{\theta}{2}\right)^2 (2\sin\theta)^{\frac{1}{2}}} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\cdot(n+2)\cdot\Gamma\left(n+\frac{3}{2}\right)}{\left(n+\frac{5}{2}\right)\Gamma\left(n+\frac{5}{2}\right)}$$

$$< \frac{4}{\pi} \cdot \frac{2 \cdot 2}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)}.$$

Introducing these approximations in (13), we obtain the asymptotic expression

$$s_n(\cos\theta) = \frac{4}{\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)} \frac{\sin\left((n+1)\theta - \frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}$$

(15)

$$+ \frac{4}{\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)} \cdot \frac{24\,\eta^*(n,\theta)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}},$$

where

$$|\eta(n,\theta)|<1 \quad \text{for} \quad 0\leqq\theta\leqq\pi \quad \text{and} \quad n=1,2,3,\cdots.$$

To obtain some information regarding the zeros of $s_n(\cos\theta)$, we write (15) in the form

$$\frac{\pi}{8}\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}\cdot 2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}\cdot s_n(\cos\theta) \tag{16}$$
$$=\sin\theta\sin\left((n+1)\theta-\frac{\pi}{4}\right)+\frac{24\eta(n,\theta)}{2n+3}.$$

Subjecting the integers n and ν to the inequalities

$$n\geqq 4^5, \tag{17}$$
$$(n+1)^{\frac{1}{2}}\leqq\nu\leqq n-(n+1)^{\frac{1}{2}},$$

we introduce the special value of θ

$$\theta_0=\frac{4\nu+1}{n+1}\frac{\pi}{4}+\frac{\kappa\pi}{(n+1)^{\frac{1}{2}}}, \qquad \kappa=\pm 1, \tag{18}$$

in the expression (16). From (18) it follows that

$$(n+1)\theta_0-\frac{\pi}{4}=\nu\pi+\frac{\kappa\pi}{(n+1)^{\frac{1}{2}}},$$

and

$$\sin\left((n+1)\theta_0-\frac{\pi}{4}\right)=(-1)^{\nu}\kappa\sin\frac{\pi}{(n+1)^{\frac{1}{2}}}.$$

Furthermore, we have, by (17) and (18),

$$0<\frac{\nu\pi}{n+1}<\theta_0<\frac{\nu+1}{n+1}\pi<\pi,$$

and, on account of $\sin x>2x/\pi$ for $0<x<\pi$, we find, in case $0<\theta_0\leqq\pi/2$,

$$\sin\theta_0>\sin\frac{\nu\pi}{n+1}>\frac{2}{\pi}\cdot\frac{\nu\pi}{n+1}\geqq\frac{2}{(n+1)^{\frac{1}{2}}},$$

and in case $\pi/2<\theta_0<\pi$,

$$\sin\theta_0>\sin\frac{\nu+1}{n+1}\pi=\sin\frac{n-\nu}{n+1}\pi>\frac{2}{\pi}\cdot\frac{n-\nu}{n+1}\pi\geqq\frac{2}{(n+1)^{\frac{1}{2}}},$$

and finally

$$\left|\sin\left((n+1)\theta_0-\frac{\pi}{4}\right)\right|=\sin\frac{\pi}{(n+1)^{\frac{1}{2}}}>\frac{2}{\pi}\cdot\frac{\pi}{(n+1)^{\frac{1}{2}}}=\frac{2}{(n+1)^{\frac{1}{2}}}.$$

From these inequalities it follows that

$$\left|\sin\theta_0 \sin\left((n+1)\theta_0 - \frac{\pi}{4}\right)\right| > \frac{2}{(n+1)^{\frac{1}{2}}} \cdot \frac{2}{(n+1)^{\frac{1}{2}}}$$

$$= \frac{4}{(n+1)^{\frac{1}{2}}} > \frac{12}{n+1} > \frac{24}{2n+3} > \left|\frac{24\eta(n,\theta_0)}{2n+3}\right|,$$

and consequently, for $\theta = \theta_0$, the expression (16) has the same sign as the first term on its right side, so that

$$\text{sgn. } s_n(\theta_0) = (-1)^\nu \cdot \kappa,$$

whence we conclude that $s_n(\theta)$ changes sign an odd number of times in the interval

$$\frac{4\nu+1}{n+1}\frac{\pi}{4} - \frac{\pi}{(n+1)^{\frac{1}{2}}} < \theta < \frac{4\nu+1}{n+1}\frac{\pi}{4} + \frac{\pi}{(n+1)^{\frac{1}{2}}}, \tag{19}$$

whenever n and ν satisfy the inequalities (17). In case the interval (19) should contain several zeros of odd order of $s_n(\theta)$, let θ_ν be the smallest among them; we then obviously have

$$\theta_\nu = \frac{4\nu+1}{n+1}\frac{\pi}{4} + \frac{h_\nu \pi}{(n+1)^{\frac{1}{2}}}, \qquad |h_\nu| < 1. \tag{20}$$

Returning to the expression (8), let $x_1, x_2, \cdots, x_m$, where $1 > x_1 > x_2 > \cdots > x_m > -1$, be the points at which $s_n(x)$ changes sign in the interval $-1 \leqq x \leqq 1$. Such points exist, on account of (20), and on the other hand, $s_n(x)$ being, by definition, a polynomial of degree n, we have $m \leqq n$.* For reasons of symmetry, we also write $x_0 = 1$ and $x_{m+1} = -1$; as $P_n(1) = 1$, $s_n(1)$ is positive, and consequently

$$|s_n(x)| = (-1)^\lambda s_n(x), \qquad x_\lambda \leqq x \leqq x_{\lambda+1}, \qquad \lambda = 0, 1, \cdots, m,$$

so that

$$\rho_n = \frac{1}{2}\int_{-1}^{1} |s_n(x)|\,dx = \left(\frac{1}{2}\int_{x_1}^{x_0} + \frac{1}{2}\int_{x_2}^{x_1} + \cdots + \frac{1}{2}\int_{x_{m+1}}^{x_m}\right)|s_n(x)|\,dx$$

$$= \frac{1}{2}\sum_{\lambda=0}^{m}(-1)^\lambda \int_{x_{\lambda+1}}^{x_\lambda} s_n(x)\,dx.$$

* We do not need to use the fact that $m = n$, which is easily proved by Christoffel's formula

$$s_n(x) = (n+1)\frac{P_n(x) - P_{n+1}(x)}{1-x}$$

and the fact that the zeros of $P_n(x)$ lie between -1 and $+1$ and separate those of $P_{n+1}(x)$. Substituting for x two consecutive roots of $P_{n+1}(x) = 0$, we obtain values of $s_n(x)$ of opposite signs, and consequently the zeros of $s_n(x)$ separate those of $P_{n+1}(x)$, or $m = n$.

Using the formula

$$s_n(x) = \frac{dP_n(x)}{dx} + \frac{dP_{n+1}(x)}{dx}, \tag{21}$$

we obtain

$$\rho_n = \frac{1}{2}\sum_{\lambda=0}^{m}(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda) - P_n(x_{\lambda+1}) - P_{n+1}(x_{\lambda+1})],$$

or rearranging the terms,

$$\rho_n = \frac{1}{2}[P_n(x_0) + P_{n+1}(x_0)] + \sum_{\lambda=1}^{m}(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda)] + \frac{(-1)^{m+1}}{2}[P_n(x_{m+1}) + P_{n+1}(x_{m+1})],$$

or finally, observing that $P_n(x_0) = 1$, $P_n(x_{m+1}) = (-1)^n$,

$$\rho_n = 1 + \sum_{\lambda=1}^{m}(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda)]. \tag{22}$$

Let ν_1 be the smallest, and ν_2 the greatest, of the integers ν satisfying (17); then to each ν, where $\nu_1 \leqq \nu \leqq \nu_2$, there corresponds, according to (20), an index λ_ν $(1 \leqq \lambda_\nu \leqq m)$ such that

$$\cos\theta_\nu = x_{\lambda_\nu}, \qquad \nu_1 \leqq \nu \leqq \nu_2.$$

We obviously have $\lambda_{\nu+1} > \lambda_\nu$, and by the definition of θ_ν, there is an even number of roots of $s_n(\cos\theta) = 0$ (or none) between θ_ν and $\theta_{\nu+1}$, so that

$$\lambda_{\nu+1} - \lambda_\nu \equiv 1 \pmod 2,$$

or

$$\lambda_{\nu+1} - (\nu+1) \equiv \lambda_\nu - \nu \pmod 2.$$

We may therefore write (22) in the form

$$\rho_n = (-1)^{\lambda_{\nu_1}-\nu_1}\sum_{\nu=\nu_1}^{\nu_2}(-1)^\nu[P_n(\cos\theta_\nu) + P_{n+1}(\cos\theta_\nu)] + 1 + \sum_{\lambda}'(-1)^\lambda[(P_n(x_\lambda) + P_{n+1}(x_\lambda)],$$

the second sum extending over such values of λ as are not of the form λ_ν, $(\nu_1 \leqq \nu \leqq \nu_2)$. As $m \leqq n$, the number of terms in Σ' does not exceed $n - (\nu_2 - \nu_1 + 1)$, which is less than or equal to $2(n+1)^{\frac{1}{2}}$, by (17). Each term in Σ' not exceeding $|P_n(x_\lambda)| + |P_{n+1}(x_\lambda)| < 1 + 1 = 2$ in absolute value, we have *

$$\left|1 + \sum_{\lambda}'(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda)]\right| < 1 + 2(n+1)^{\frac{1}{2}} \cdot 2 = O(n^{\frac{1}{2}}),$$

* The notation $f(n) = O(g(n))$ signifies that for n sufficiently large, $|f(n)| < Ag(n)$, where A is a constant independent of n. Similarly, $f(n) = o(g(n))$ stands for $\lim_{n=\infty}\frac{f(n)}{g(n)} = 0$.

and consequently

$$\rho_n = (-1)^{\lambda_{\nu_1}-\nu_1} \sum_{\nu=\nu_1}^{\nu_2} (-1)^\nu [P_n(\cos\theta_\nu) + P_{n+1}(\cos\theta_\nu)] + O(n^{\frac{1}{2}}). \tag{23}$$

Our next step will obviously be the asymptotic determination of the general term in the series above, and to this purpose we use the following formula, due to Stieltjes* and constituting a generalization of the well-known asymptotic formula of Laplace:

$$\begin{aligned} P_n(\cos\theta) = \frac{2}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \Bigg[& \frac{\cos\left(\frac{2n+1}{2}\theta - \frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}} \\ & + \frac{1^2}{2(2n+3)} \cdot \frac{\cos\left(\frac{2n+3}{2} - \frac{3\pi}{4}\right)}{(2\sin\theta)^{\frac{3}{2}}} + \cdots \\ & + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2p-3)^2}{2 \cdot 4 \cdots (2p-2)(2n+3)\cdots(2n+2p-1)} \frac{\cos\left(\frac{2n+2p-1}{2}\theta - \frac{2p-1}{4}\pi\right)}{(2\sin\theta)^{\frac{2p-1}{2}}} \\ & + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2p-1)^2}{2 \cdot 4 \cdots 2p(2n+3)(2n+5)\cdots(2n+2p+1)} \frac{M(p,n,\theta)}{(2\sin\theta)^{\frac{2p+1}{2}}} \Bigg], \end{aligned} \tag{24}$$

where

$$|M(p,n,\theta)| < 2 \quad \text{for} \quad 0 \leqq \theta \leqq \pi.$$

For our purpose, it is sufficiently accurate to make $p = 1$ in (24), and we obtain

$$\begin{aligned} P_n(\cos\theta) + P_{n+1}(\cos\theta) = \frac{2}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \Bigg\{ & \frac{2\cos\frac{\theta}{2}\cos\left((n+1)\theta - \frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}} \\ & + \frac{1}{(2n+3)(2\sin\theta)^{\frac{3}{2}}} \Bigg[-2\sin\theta\cos\left(\frac{2n+3}{2}\theta - \frac{\pi}{4}\right) \\ & + \frac{M(1,n,\theta)}{2} + \frac{2n+2}{2n+5} \frac{M(1,n+1,\theta)}{2} \Bigg] \Bigg\} \\ = \frac{2}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \Bigg\{ & \cos\left((n+1)\theta - \frac{\pi}{4}\right)\sqrt{\cot\frac{\theta}{2}} \\ & + \frac{M'(n,\theta)}{(2n+3)(2\sin\theta)^{\frac{3}{2}}} \Bigg\}, \end{aligned} \tag{25}$$

*T. J. Stieltjes, *Sur les polynômes de Legendre*, Annales de la Faculté des Sciences de Toulouse, ser. I, vol. 4 (1890), pp. *G*1 – *G*17. The method used above for arriving at the asymptotic expression for $s_n(\cos\theta)$ was suggested by Stieltjes' proof of (24).

where

$$|M'(n,\theta)| < 4 \quad \text{for} \quad 0 \leqq \theta \leqq \pi.$$

By (20) and the various inequalities following (18), we have

$$\cos\left((n+1)\theta_\nu - \frac{\pi}{4}\right) = (-1)^\nu \cos\frac{h_\nu \pi}{(n+1)^{\frac{1}{2}}}$$

$$= (-1)^\nu\left[1 + O\left(\frac{h_\nu^2 \pi^2}{(n+1)^{\frac{1}{2}}}\right)\right] = (-1)^\nu\left[1 + O\left(\frac{1}{n^{\frac{1}{2}}}\right)\right],$$

$$\cot\frac{\theta_\nu}{2} < \frac{1}{\sin\frac{\theta_\nu}{2}} < \frac{1}{\frac{2}{\pi}\cdot\frac{\nu\pi}{2(n+1)}} \leqq (n+1)^{\frac{1}{2}};$$

consequently

$$\cos\left((n+1)\theta_\nu - \frac{\pi}{4}\right)\sqrt{\cot\frac{\theta_\nu}{2}} = (-1)^\nu\sqrt{\cot\frac{\theta_\nu}{2}} + O\left(\frac{1}{n^{\frac{1}{10}}}\right),$$

and furthermore

$$\frac{M'(n,\theta_\nu)}{(2n+3)(2\sin\theta_\nu)^{\frac{3}{2}}} = O\left(\frac{1}{2n+3}\cdot(n+1)^{\frac{9}{10}}\right) = O\left(\frac{1}{n^{\frac{1}{10}}}\right);$$

finally Stirling's formula,

$$\log\Gamma(x+1) = (x+\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log 2\pi + \sum_{\nu=1}^{p}\frac{(-1)^{\nu-1}B_\nu}{(2\nu-1)\cdot 2\nu}\cdot\frac{1}{x^{2\nu-1}}$$

$$+ \frac{(-1)^p B_{p+1}}{(2p+1)(2p+2)}\frac{\vartheta}{x^{2p+1}}, \qquad 0 < \vartheta < 1,$$

gives, for $p = 1$, and since $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$\frac{2}{\pi}\frac{\Gamma(\frac{1}{2})\,\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} = \frac{2}{\sqrt{\pi n}}\left[1 - \frac{1}{4n} + O\left(\frac{1}{n^2}\right)\right]. \tag{26}$$

Introducing all this in (25), we obtain

$$(-1)^\nu[P_n(\cos\theta_\nu) + P_{n+1}(\cos\theta_\nu)] = \frac{2}{\sqrt{\pi n}}\sqrt{\cot\frac{\theta_\nu}{2}} + O\left(\frac{1}{n^{\frac{1}{2}}}\right),$$

whence, by (23),

$$(-1)^{\lambda_{\nu_1}-\nu_1}\rho_n = \frac{2}{\sqrt{\pi n}}\sum_{\nu=\nu_1}^{\nu_2}\sqrt{\cot\frac{\theta_\nu}{2}} + \sum_{\nu=\nu_1}^{\nu_2} O\left(\frac{1}{n^{\frac{1}{2}}}\right) + O(n^{\frac{1}{2}}).$$

In the second sum, each term is less than a constant multiple of $n^{-\frac{1}{2}}$, and the number of terms is less than n, so that

$$\sum_{\nu=\nu_1}^{\nu_2} O\left(\frac{1}{n^{\frac{1}{2}}}\right) = O\left(n\cdot\frac{1}{n^{\frac{1}{2}}}\right) = O(n^{\frac{1}{2}}),$$

and consequently

$$(-1)^{\lambda_{\nu_1}-\nu_1}\rho_n = \frac{2}{\sqrt{\pi n}}\sum_{\nu=\nu_1}^{\nu_2}\sqrt{\cot\frac{\theta_\nu}{2}} + O(n^{\frac{1}{2}}),$$

which may be written

$$(-1)^{\lambda_{\nu_1}-\nu_1}\frac{\rho_n}{\sqrt{n}} = \frac{2}{\pi\sqrt{\pi}}\cdot\frac{n+1}{n}\sum_{\nu=\nu_1}^{\nu_2}\left(\frac{4\nu+3}{n+1}\frac{\pi}{4} - \frac{4\nu-1}{n+1}\frac{\pi}{4}\right)\sqrt{\cot\frac{\theta_\nu}{2}} + O\left(\frac{1}{n^{\frac{1}{10}}}\right),$$

By (17) and (20), we have

$$\frac{4\nu+3}{n+1}\frac{\pi}{4} > \theta_\nu > \frac{4\nu-1}{n+1}\frac{\pi}{4},$$

and the preceding equation gives, when we increase n indefinitely,*

$$\lim_{n=\infty}(-1)^{\lambda_{\nu_1}-\nu_1}\frac{\rho_n}{\sqrt{n}} = \frac{2}{\pi\sqrt{\pi}}\int_0^\pi\sqrt{\cot\frac{\theta}{2}}\,d\theta.$$

From the equation of definition (8), it follows that ρ_n is positive, and consequently

$$(-1)^{\lambda_{\nu_1}-\nu_1} = 1$$

for all sufficiently large values of n, so that

$$\lim_{n=\infty}\frac{\rho_n}{\sqrt{n}} = \frac{2}{\pi\sqrt{\pi}}\int_0^\pi\sqrt{\cot\frac{\theta}{2}}\,d\theta.$$

Making the substitution $\cot\theta/2 = u^2$, we obtain

$$\int_0^\pi\sqrt{\cot\frac{\theta}{2}}\,d\theta = \int_0^\infty\frac{4u^2\,du}{1+u^4} = \int_{-\infty}^{+\infty}\frac{2u^2\,du}{1+u^4},$$

and by a classical application of Cauchy's theorem, the latter integral equals $2\pi i$ times the sum of the residues of the integrand in the upper half plane, or

$$2\pi i\left\{\left(\frac{2u^2}{4u^3}\right)_{u=\frac{1+i}{\sqrt{2}}} + \left(\frac{2u^2}{4u^3}\right)_{u=\frac{-1+i}{\sqrt{2}}}\right\} = \pi i\sqrt{2}\left(\frac{1}{1+i}+\frac{1}{-1+i}\right) = \pi\sqrt{2},$$

so that finally,

$$\lim_{n=\infty}\frac{\rho_n}{\sqrt{n}} = 2\sqrt{\frac{2}{\pi}}. \tag{27}$$

§3. Approximation to functions satisfying a generalized Lipschitz condition by finite sums of spherical harmonics.

THEOREM I. *Let $f(\theta,\varphi)$ be a uniform function of the position on the sphere S, satisfying the generalized Lipschitz condition for any two points θ,φ and θ',φ':*

$$|f(\theta',\varphi') - f(\theta,\varphi)| \leqq \omega(\gamma), \tag{28}$$

* The integrand decreasing monotonously as θ increases, this passage to the limit is seen at once to be legitimate, although the integral obtained is an improper one.

*the function $\omega(\gamma)$ being subject to the conditions**

$$\omega(\gamma) \leqq \omega(\gamma'), \qquad \frac{\omega(\gamma)}{\gamma} \geqq \frac{\omega(\gamma')}{\gamma'} \quad \text{for} \quad 0 < \gamma \leqq \gamma' < \pi; \tag{29}$$
$$\lim_{\gamma=0} \omega(\gamma) = 0.$$

Then there exists, for every positive integer n, a sum $T_n(\theta, \varphi)$, of a finite number of spherical harmonics of order $\overline{\leqq} n$, approximating $f(\theta, \varphi)$ in such a way that

$$|f(\theta, \varphi) - T_n(\theta, \varphi)| \overline{\leqq} K\omega(1/n)$$

at every point of the sphere S, K being a constant independent of n.

In order to prove this theorem, we shall use a special type of functions $T_n(\theta, \varphi)$, defined by †

$$T_n(\theta, \varphi) = \frac{\int_S f(\theta', \varphi') \left[\dfrac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}{\int_S \left[\dfrac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}, \tag{30}$$

where the integer m is related to n by

$$2(m-1) \overline{\leqq} n < 2m. \tag{31}$$

The function $T_n(\theta, \varphi)$ is actually a sum of a finite number of spherical harmonics of order $\leqq n$; to prove this assertion, we start from the relation

$$\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}} = \frac{e^{\frac{m\gamma i}{2}} - e^{-\frac{m\gamma i}{2}}}{e^{\frac{\gamma i}{2}} - e^{-\frac{\gamma i}{2}}} = e^{-(m-1)\frac{\gamma i}{2}} \cdot \frac{1 - e^{m\gamma i}}{1 - e^{\gamma i}} = e^{-(m-1)\frac{\gamma i}{2}} \sum_{\nu=0}^{m-1} e^{\nu\gamma i}, \tag{32}$$

whence

$$\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 = e^{-2(m-1)\gamma i} \sum_{\nu_1=0}^{m-1}\sum_{\nu_2=0}^{m-1}\sum_{\nu_3=0}^{m-1}\sum_{\nu_4=0}^{m-1} e^{(\nu_1+\nu_2+\nu_3+\nu_4)\gamma i} = \sum_{\lambda=-2(m-1)}^{+2(m-1)} c_\lambda e^{\lambda\gamma i}.$$

Changing γ and λ into $-\gamma$ and $-\lambda$, we see at once that $c_\lambda = c_{-\lambda}$ and the

* From the last condition, it follows that $f(\theta, \varphi)$ is continuous in every point of S.

† The corresponding finite trigonometric sums for a function $f(x)$ satisfying the ordinary Lipschitz condition $|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|$, λ = const., were first introduced by Jackson (see papers II and III, quoted in § 1).

last equation becomes

$$(33)\qquad \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 = c_0 + \sum_{\lambda=1}^{2(m-1)} c_\lambda (e^{\lambda\gamma i} + e^{-\lambda\gamma i}) = c_0 + 2\sum_{\lambda=1}^{2(m-1)} c_\lambda \cos\lambda\gamma .$$

Expressing $\cos\lambda\gamma$ as a polynomial of degree λ in $\cos\gamma$, and this polynomial being expressed by the Legendre polynomials in $\cos\gamma$ of degree $\leqq\lambda$, we reduce (30) to the form

$$T_n(\theta,\varphi) = \sum_{\nu=0}^{2(m-1)} a_\nu \int_S f(\theta',\varphi')\, P_\nu(\cos\gamma)\, d\sigma' ,$$

the general term on the right side being obviously a spherical harmonic of order $\nu \leqq 2(m-1) \leqq n$.

Having established this point, we may write (30) in the form

$$f(\theta,\varphi) - T_n(\theta,\varphi) = \frac{\int_S [f(\theta,\varphi) - f(\theta',\varphi')] \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}{\int_S \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'},$$

and by (28) we obtain

$$(34)\qquad |f(\theta,\varphi) - T_n(\theta,\varphi)| \leqq \frac{\int_S \omega(\gamma) \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}{\int_S \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'} .$$

In the integrals, we now move the north pole to the point θ, φ; then $\gamma = \theta'$ and $d\sigma' = \sin\gamma\, d\gamma d\varphi$, so that

$$\int_S \omega(\gamma) \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma' = \int_0^\pi \int_0^{2\pi} \omega(\gamma) \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma\, d\varphi$$

$$= 2\pi \int_0^\pi \omega(\gamma) \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma ,$$

$$\int_s \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma' = 2\pi\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma.$$

By the aid of (29), we obtain

$$\int_0^\pi \omega(\gamma)\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma = \int_0^{1/n} + \int_{1/n}^\pi \leqq \int_0^{1/n} \omega\left(\frac{1}{n}\right)\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma$$

$$+ \int_{1/n}^\pi \frac{\omega\left(\frac{1}{n}\right)}{\frac{1}{n}}\cdot\gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma$$

$$< \omega\left(\frac{1}{n}\right)\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma + n\omega\left(\frac{1}{n}\right)\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma$$

$$< \omega\left(\frac{1}{n}\right)\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma + 2m\omega\left(\frac{1}{n}\right)\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma.$$

nce $n < 2m$, by (31), it follows from (34) that

$$(35)\qquad |f(\theta,\varphi) - T_n(\theta,\varphi)| < \omega\left(\frac{1}{n}\right)\left[1 + \frac{2m\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma}{\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma}\right].$$

Observing that $\sin x > 2x/\pi$ for $0 < x < \pi/2$ and $\sin x < x$ for $0 < x < \pi$, we find

$$2m\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma < 2m\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\frac{\gamma}{\pi}}\right]^4 \gamma\, d\gamma = 2\pi^4 m\int_0^\pi \frac{\sin^4\frac{m\gamma}{2}}{\gamma^2}\, d\gamma$$

$$= \pi^4 m^2\int_0^{\frac{m\pi}{2}} \frac{\sin^4\gamma}{\gamma^2}\, d\gamma < \pi^4 m^2\int_0^\infty \frac{\sin^4\gamma}{\gamma^2}\, d\gamma,$$

$$\int_0^{\pi}\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4\sin\gamma\,d\gamma>\int_0^{\frac{\pi}{2}}\left[\frac{\sin\frac{u\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4\sin\gamma\,d\gamma>\int_0^{\frac{\pi}{2}}\left[\frac{\sin\frac{m\gamma}{2}}{\frac{\gamma}{2}}\right]^4\cdot\frac{2}{\pi}\gamma\,d\gamma$$

$$=\frac{32}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin^4\frac{m\gamma}{2}}{\gamma^3}\,d\gamma=\frac{8}{\pi}m^2\int_0^{\frac{m\pi}{4}}\frac{\sin^4\gamma}{\gamma^3}\,d\gamma\geqq\frac{8}{\pi}m^2\int_0^{\frac{\pi}{4}}\frac{\sin^4\gamma}{\gamma^3}\,d\gamma,$$

and from (34) we obtain

$$|f(\theta,\varphi)-T_n(\theta,\varphi)|<\omega\left(\frac{1}{n}\right)\left\{1+\frac{\pi^5\int_0^{\infty}\frac{\sin^4\gamma}{\gamma^2}\,d\gamma}{8\int_0^{\frac{\pi}{4}}\frac{\sin^4\gamma}{\gamma^3}\,d\gamma}\right\}\quad(n=1,2,3,\cdots),$$

which proves our theorem.

Although not necessary for the application in § 4, it is interesting to derive from (35) a fairly small numerical upper limit for the constant K in theorem I. With the notations

$$I_m=\frac{1}{4m^2}\int_0^{\pi}\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4\sin\gamma\,d\gamma,\qquad I'_m=\frac{1}{2m}\int_0^{\pi}\gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4\sin\gamma\,d\gamma$$

equation (35) may be written

$$|f(\theta,\varphi)-T_n(\theta,\varphi)|<\left(1+\frac{I'_m}{I_m}\right)\omega\left(\frac{1}{n}\right),\tag{36}$$

and making $\gamma=2u$ and using the inequality $u\cos u<\sin u\left(0<u<\frac{\pi}{2}\right)$, we have

$$I_m=\frac{1}{m^2}\int_0^{\frac{\pi}{2}}\frac{\sin^4 mu\cos u}{\sin^3 u}\,du,$$

$$I'_m=\frac{4}{m}\int_0^{\frac{\pi}{2}}\frac{u\sin^4 mu\cos u}{\sin^3 u}\,du<\frac{4}{m}\int_0^{\frac{\pi}{2}}\frac{\sin^4 mu}{\sin^2 u}\,du.\tag{37}$$

Considering first the integral I_m, we integrate by parts and obtain

$$m^2 I_m=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^4 mu\cdot d\left(\frac{1}{\sin^2 u}\right)=-\frac{1}{2}\sin^4\frac{m\pi}{2}+2m\int_0^{\frac{\pi}{2}}\frac{\sin^3 mu\cos mu}{\sin^2 u}\,du$$

$$=\frac{(-1)^m-1}{4}-2m\int_0^{\frac{\pi}{2}}\sin^3 mu\cos mu\cdot d\cot u$$

$$=\frac{(-1)^m-1}{4}+2m^2\int_0^{\frac{\pi}{2}}(3\sin^2 mu\cos^2 mu-\sin^4 mu)\cot u\,du;$$

as $3\sin^2 mu\cos^2 mu - \sin^4 mu = \frac{1}{2}(\cos 2mu - \cos 4mu)$, this gives

$$I_m = \frac{(-1)^m - 1}{4m^2} + j_m,$$

where

$$j_m = \int_0^{\frac{\pi}{2}} \cot u\,(\cos 2mu - \cos 4mu)\,du.$$

Consider the difference $j_m - j_{m-1}$; the identities

$$\begin{aligned}
&\cot u\,[\cos 2mu - \cos 4mu - (\cos 2(m-1)u - \cos 4(m-1)u)]\\
&\quad = \cot u\,[-2\sin u \sin(2m-1)u + 2\sin 2u \sin(4m-2)u]\\
&\quad = -2\cos u \sin(2m-1)u + 4\cos^2 u \sin(4m-2)u\\
&\quad = -2\cos u \sin(2m-1)u + 2(1+\cos 2u)\sin(4m-2)u\\
&\quad = -\sin 2mu - \sin(2m-2)u + 2\sin(4m-2)u + \sin 4mu\\
&\qquad\qquad + \sin(4m-4)u
\end{aligned}$$

give, upon integration between the limits 0 and $\pi/2$,

$$j_m - j_{m-1} = \frac{1}{2m}[(-1)^m - 1] + \frac{1}{2m-2}[(-1)^{m-1} - 1] + \frac{2}{2m-1},$$

where, for $m = 1$, the second term on the right side should be replaced by zero. On account of $j_0 = 0$, we conclude that

$$\begin{aligned}
j_m &= \sum_{\nu=1}^{m}\left[\frac{1}{2\nu}[(-1)^\nu - 1] + \frac{1}{2\nu-2}[(-1)^{\nu-1} - 1] + \frac{2}{2\nu-1}\right]\\
&= \frac{(-1)^m - 1}{2m} + \frac{2}{2m-1} + \sum_{\nu=1}^{m-1}\left[\frac{(-1)^\nu - 1}{\nu} + \frac{2}{2\nu-1}\right],
\end{aligned}$$

the result being also true for $m = 1$, if the sum on the right side is replaced by zero. It is now seen that

$$(38)\quad I_m = \sum_{\nu=1}^{m-1}\left(\frac{(-1)^\nu - 1}{\nu} + \frac{2}{2\nu-1}\right) + \frac{(-1)^m - 1}{2m} + \frac{2}{2m-1} + \frac{(-1)^m - 1}{4m^2},$$

and

$$I_{m+1} - I_m = \frac{(-1)^m - 1}{2m} - \frac{(-1)^m - 1}{4m^2} - \frac{(-1)^m + 1}{2(m+1)} + \frac{2}{2m+1} - \frac{(-1)^m + 1}{4(m+1)^2}$$

or, when the cases m odd and m even are separated,

$$I_{m+1} - I_m = \begin{cases} \dfrac{1}{2m^2(2m+1)}, & m \text{ odd},\\[2ex] \dfrac{1}{2(m+1)^2(2m+1)}, & m \text{ even}, \end{cases}$$

so that in both cases

(39) $$I_{m+1} > I_m .$$

For $m = 2$, (38) gives the value $2/3$, whence by (39)

(40) $$I_m \gtreqless \tfrac{2}{3} = 0.6667 - \quad \text{for} \quad m \gtreqless 2 .$$

To form an idea of the approximation furnished by (40), we observe that by equations (39) and (38),

$$I_m < I_{2m} < \lim_{m=\infty} I_{2m} = \lim_{m=\infty} \sum_{\nu=1}^{2m-1} \left(\frac{(-1)^\nu - 1}{\nu} + \frac{2}{2\nu - 1} \right)$$

$$= \lim_{m=\infty} \left(\sum_{\nu=1}^{2m-1} \frac{2}{2\nu - 1} - \sum_{\nu=1}^{m} \frac{2}{2\nu - 1} \right) = \lim_{m=\infty} \sum_{\nu=m+1}^{2m-1} \frac{2}{2\nu - 1}$$

$$= \lim_{m=\infty} \frac{1}{2m-1} \sum_{\nu=1}^{m-1} \frac{1}{\frac{1}{2} + \frac{\nu}{2m-1}} = \int_{\frac{1}{2}}^{1} \frac{dx}{x}$$

$$= \log 2 = 0.6931 + .$$

Passing to the last integral in (37), we have

$$\frac{4}{m} \int_0^{\frac{\pi}{2}} \frac{\sin^4 mu}{\sin^2 u}\, du = \frac{2}{m} \int_0^{\pi} (1 - \cos 2mu) \left(\frac{\sin mu}{\sin u} \right)^2 du .$$

Writing $\gamma = 2u$ in (32) and squaring, we obtain

$$\left(\frac{\sin mu}{\sin u} \right)^2 = e^{-2(m-1)ui} \sum_{\nu_1=0}^{m-1} \sum_{\nu_2=0}^{m-1} e^{2(\nu_1+\nu_2)ui} = \sum_{\lambda=-(m-1)}^{+(m-1)} c'_\lambda e^{2\lambda ui};$$

changing u and λ into $-u$ and $-\lambda$, we find $c'_\lambda = c'_{-\lambda}$, so that

$$\left(\frac{\sin mu}{\sin u} \right)^2 = c'_0 + 2 \sum_{\lambda=1}^{m-1} c'_\lambda \cos 2\lambda u ,$$

whence

$$\frac{4}{m} \int_0^{\frac{\pi}{2}} \frac{\sin^4 mu}{\sin^2 u}\, du = \frac{2}{m} \int_0^{\pi} (1 - \cos 2mu) \left(c'_0 + 2 \sum_{\lambda=1}^{m-1} c'_\lambda \cos 2\lambda u \right) du$$

$$= \frac{2}{m} \cdot \frac{\pi}{2} c'_0 .$$

From (32) it is seen immediately that c'_0 is the constant term in the expansion of

$$x^{-(m-1)} \left(\frac{1 - x^m}{1 - x} \right)^2 = x^{-(m-1)} (1 - 2x^m + x^{2m}) \sum_{\nu=0}^{\infty} (\nu + 1)\, x^\nu ,$$

or $c'_0 = m$, so that we obtain from (37)

(41) $$I'_m < \pi .$$

Equations (40) and (41) now give

$$1+\frac{I'_m}{I_m}<1+\frac{3\pi}{2}<6 \quad \text{for} \quad m \gtreqless 2;$$

that is, on account of (31), for $n \gtreqless 2$, and from (36) we derive the desired result

$$|f(\theta, \varphi)-T_n(\theta, \varphi)|<6\omega\left(\frac{1}{n}\right) \quad (n=2, 3, 4, \cdots). \tag{42}$$

Before leaving this subject, we shall evaluate the integral

$$J_m=\frac{1}{m^3}\int_0^{\frac{\pi}{2}}\left(\frac{\sin mu}{\sin u}\right)^4 du \tag{43}$$

occurring in Jackson's investigation of the approximation to a function $f(x)$ of period 2π and satisfying a Lipschitz condition

$$f(x_2)-f(x_1)|\leqq\lambda|x_2-x_1|$$

by finite trigonometric sums. By an asymptotic process, Jackson* shows that

$$J_m>J_{m+1}, \qquad \lim_{m=\infty} J_m=\frac{\pi}{3}. \tag{44}$$

Writing $\gamma=2u$, we obtain at once from (33)

$$J_m=\frac{c_0}{m^3}\cdot\frac{\pi}{2}.$$

By (32) and (33), c_0 is obviously the constant term in the expansion of

$$x^{-2(m-1)}\left(\frac{1-x^m}{1-x}\right)^4=x^{-2(m-1)}(1-4x^m+6x^{2m}-4x^{3m}+x^{4m})\times\sum_{\nu=0}^{\infty}\frac{(\nu+1)(\nu+2)(\nu+3)}{1\cdot 2\cdot 3}x^\nu,$$

so that, the terms in the infinite series with $\nu=2(m-1)$ and $\nu=m-2$ being the only ones contributing to c_0,

$$c_0=\frac{(2m-1)\cdot 2m(2m+1)}{1\cdot 2\cdot 3}-4\cdot\frac{(m-1)\cdot m(m+1)}{1\cdot 2\cdot 3}=\frac{m}{3}(2m^2+1),$$

and consequently

$$J_m=\frac{\pi}{3}\left(1+\frac{1}{2m^2}\right). \tag{45}$$

From this expression, the two properties (44) follow at once.

* In the paper previously quoted as III, pp. 503–507.

§ 4. The degree of convergence of Laplace's series for functions satisfying a generalized Lipschitz condition.

THEOREM II. *When $f(\theta, \varphi)$ satisfies the generalized Lipschitz condition*

$$|f(\theta', \varphi') - f(\theta, \varphi)| \leqq \omega(\gamma) \tag{28}$$

for any two points θ, φ and θ', φ' on the sphere S, the function $\omega(\gamma)$ being subject to the conditions

$$\omega(\gamma) \leqq \omega(\gamma'), \quad \frac{\omega(\gamma)}{\gamma} \geqq \frac{\omega(\gamma')}{\gamma'} \text{ for } 0 < \gamma \leqq \gamma' < \pi, \quad \lim_{\gamma=0} \omega(\gamma) = 0, \tag{29}$$

then $f(\theta, \varphi)$ is approximated by the first $n+1$ terms of its Laplace series in such a way that the remainder

$$r_n\{f(\theta, \varphi)\} = f(\theta, \varphi) - s_n\{f(\theta, \varphi)\} = f(\theta, \varphi) - \frac{1}{4\pi}\int_S f(\theta', \varphi')\, s_n(\cos\gamma)\, d\sigma'$$

satisfies the inequality

$$|r_n\{f(\theta, \varphi)\}| \leqq K'\omega\left(\frac{1}{n}\right)\sqrt{n} \qquad (n = 1, 2, 3, \cdots),$$

K' being a constant independent of n.

The principle of the following proof is due to Lebesgue, in the case of Fourier's series.* From the identity

$$f(\theta, \varphi) = T_n(\theta, \varphi) + [f(\theta, \varphi) - T_n(\theta, \varphi)]$$

and the definition of the remainder r_n, it follows that

$$r_n\{f(\theta, \varphi)\} = r_n\{T_n(\theta, \varphi)\} + r_n\{f(\theta, \varphi) - T_n(\theta, \varphi)\};$$

the Laplace series of $T_n(\theta, \varphi)$ being obviously this function itself, it is seen that

$$r_n\{T_n(\theta, \varphi)\} = 0.$$

The definition of r_n then gives

$$|r_n\{f(\theta, \varphi)| = |r_n\{f(\theta, \varphi) - T_n(\theta, \varphi)\}| \leqq |f(\theta, \varphi) - T_n(\theta, \varphi)| + \frac{1}{4\pi}\int_S |f(\theta', \varphi') - T_n(\theta', \varphi')|\, |s_n(\cos\gamma)|\, d\sigma',$$

and if we apply theorem I,

$$|r_n\{f(\theta, \varphi)\}| \leqq K\omega\left(\frac{1}{n}\right) + K\omega\left(\frac{1}{n}\right)\cdot\frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\, d\sigma' = K\omega\left(\frac{1}{n}\right)(1 + \rho_n),$$

* H. Lebesgue, *Sur la représentation trigonométrique des fonctions, etc.*, Bulletin de la Société Mathématique de France, vol. 38 (1910), pp. 184–210; pp. 196–197; *Sur les intégrales singulières*, Annales de la Faculté des Sciences de Toulouse, ser. 3, vol. 1 (1910), pp. 25–117; pp. 116–117. Compare also Jackson, I, II, III.

by the definition of ρ_n at the beginning of § 2. On account of (27), there exists a constant c independent of n and such that

$$\rho_n \leqq c\sqrt{n} \qquad (n = 1, 2, 3, \cdots).$$

The preceding inequality now shows that

$$|r_n\{f(\theta, \varphi)\}| \leqq K(1+c\sqrt{n})\omega\left(\frac{1}{n}\right) \leqq K(1+c)\,\omega\left(\frac{1}{n}\right)\sqrt{n} = K'\omega\left(\frac{1}{n}\right)\sqrt{n},$$

which proves our theorem.

It follows, in particular, from theorem II that *whenever*

$$\lim_{n=\infty} \omega\left(\frac{1}{n}\right)\sqrt{n} = 0,$$

the Laplace series corresponding to $f(\theta, \varphi)$ *is convergent and equal to this function at every point of the sphere* S.

In the particular case of the Legendre series corresponding to a function $f(x) = f(\cos\theta)$ satisfying the condition (5), which is analogous to (28), Jackson* has shown that for any positive ϵ there exists a constant $K'(\epsilon)$ independent of n (but depending, of course, on ϵ) such that

$$|\,r_n\{f(x)\}\,| < K'(\epsilon)\,\omega\left(\frac{1}{n}\right)\log n \qquad (n = 2, 3, 4, \cdots),$$

uniformly in the interval

$$-1+\epsilon \leqq x \leqq 1-\epsilon.$$

This result cannot be extended to the limiting case $\epsilon = 0$; on the contrary, we have the following theorem, which may be considered as the reciprocal of theorem II:

THEOREM III. *Let* $\omega(\gamma)$ *be a function subject to the conditions*

$$(29) \quad \omega(\gamma) \leqq \omega(\gamma'), \quad \frac{\omega(\gamma)}{\gamma} \geqq \frac{\omega(\gamma')}{\gamma'} \quad \textit{for } 0 < \gamma \leqq \gamma' < \pi; \quad \lim_{\gamma=0} \omega(\gamma) = 0,$$

and $\Omega(\gamma)$ *any other function such that*

$$(46) \qquad \lim_{\gamma=0} \frac{\Omega(\gamma)}{\omega(\gamma)} = 0;$$

there exists a function $f(\theta, \varphi)$ *satisfying the generalized Lipschitz condition*

$$(28) \qquad |f(\theta', \varphi') - f(\theta, \varphi)| \leqq \omega(\gamma)$$

for any two points θ, φ *and* θ', φ' *on the sphere* S, *and such that, at a given point*

* Paper referred to as I, theorem (1).

θ_0, φ_0, the remainder in its Laplace series satisfies the inequality

$$|\, r_n \{ f(\theta_0, \varphi_0) \} \,| \geqq K'' \, \Omega\left(\frac{1}{n}\right) \sqrt{n}$$

for an infinity of values of n, the constant K'' being independent of n.

To prove this theorem, we shall construct a special function $f(\theta, \varphi)$ having the required properties,* choosing the given point at $\theta_0 = 0$, $\varphi_0 = 0$, and making $f(\theta, \varphi) = F(\theta)$ independent of φ, so that our Laplace series will reduce to a Legendre series. Our function $F(\theta)$ will be built up from elements of the form $F_n(\theta)$, this latter function being defined by

$$(47) \quad F_n(\theta) = \sin\left((n+1)\,\theta - \frac{\pi}{4}\right) \quad \text{for} \quad \frac{1}{n+1}\,\frac{\pi}{4} \leqq \theta \leqq \frac{4\mu+1}{n+1}\,\frac{\pi}{4},$$

$$= 0 \text{ outside of this interval,}$$

μ and n being positive integers (or zero) to be determined later. $F_n(\theta)$ is continuous, as it vanishes at the end points of the interval above, which we may denote by (μ, n).

By (47), we have $F_n(\theta) = 0$ for $\theta = 0$, so that, developing $F_n(\theta)$ in a Laplace series, we find

$$r_n \{ F_n(0) \} = - s_n \{ F_n(0) \} = - \frac{1}{4\pi} \int_S F_n(\theta')\, s_n(\cos\gamma)\, d\sigma',$$

or, introducing $d\sigma' = \sin\theta'\, d\theta'\, d\varphi'$ and using the well-known formula

$$\int_0^{2\pi} P_n(\cos\gamma)\, d\varphi' = 2\pi\, P_n(\cos\theta)\, P_n(\cos\theta'),$$

making $\theta = 0$, so that $P_n(\cos\theta) = 1$, and finally writing θ instead of θ',

$$r_n \{ F_n(0) \} = - \frac{1}{2} \int_0^{\pi} F_n(\theta)\, s_n(\cos\theta) \sin\theta\, d\theta.$$

Since $F_n(\theta)$ is identically equal to zero in part of the interval of integration, we may write

$$r_n \{ F_n(0) \} = - \frac{1}{2} \int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} F_n(\theta)\, s_n(\cos\theta) \sin\theta\, d\theta;$$

or introducing the value $F_n(\theta) = \sin\left((n+1)\,\theta - \frac{\pi}{4}\right)$ in the interval in question, and using (15),

* In principle, the method of proof is due to Lebesgue, who applied it to Fourier's series; see papers by Lebesgue and Jackson previously quoted.

$$r_n\{F_n(0)\} = -\frac{2}{\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)}\times$$

(48)
$$\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}}\left[\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}}\cos\frac{\theta}{2}+\frac{24\eta(n,\theta)}{2n+3}\frac{\sin\left((n+1)\theta-\frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}\right]d\theta.$$

Now assume $\mu \leqq \frac{n}{2}$; we then have, in the interval of integration in (48)

(49) $$\cos\frac{\theta}{2} > \frac{1}{\sqrt{2}}, \qquad \theta > \sin\theta > \frac{2}{\pi}\theta, \qquad 2\sin\frac{\theta}{2} > \frac{2}{\pi}\theta,$$

so that

$$\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}}\cos\frac{\theta}{2}\,d\theta > \frac{1}{2}\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{\sqrt{\theta}}\,d\theta$$

$$= \frac{1}{2}\sum_{\nu=1}^{\mu}\int_{\frac{4\nu-3}{n+1}\frac{\pi}{4}}^{\frac{4\nu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{\sqrt{\theta}}\,d\theta$$

(50)
$$> \frac{1}{2}\sum_{\nu=1}^{\mu}\int_{\frac{4\nu-3}{n+1}\frac{\pi}{4}}^{\frac{4\nu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{\sqrt{\frac{4\nu+1}{n+1}\frac{\pi}{4}}}\,d\theta$$

$$= \frac{\sqrt{\pi}}{8}\cdot\frac{1}{\sqrt{n+1}}\sum_{\nu=1}^{\mu}\frac{1}{\sqrt{4\nu+1}}$$

$$> \frac{\sqrt{\pi}}{8}\cdot\frac{1}{\sqrt{n+1}}\int_{1}^{\mu+1}\frac{du}{\sqrt{4u+1}}$$

$$= \frac{\sqrt{\pi}}{16}\cdot\frac{1}{\sqrt{n+1}}\left(\sqrt{4(\mu+1)+1}-3\right).$$

For the second part of the integral in (48) we obtain, using (49) and observing that $|\eta(n,\theta)| < 1$,

$$\left|\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \frac{24\eta(n,\theta)}{2n+3}\,\frac{\sin\left((n+1)\theta-\frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}\,d\theta\right| < \int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \frac{24}{2n+2}\cdot\frac{1}{\frac{2}{\pi}\theta\cdot\left(\frac{4}{\pi}\theta\right)^{\frac{1}{2}}}\,d\theta$$

51)

$$= \frac{3\pi^{\frac{1}{2}}}{n+1}\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \frac{d\theta}{\theta^{\frac{1}{2}}}$$

$$= \frac{12\pi}{\sqrt{n+1}}\left(1-\frac{1}{\sqrt{4\mu+1}}\right);$$

and on combining (50) with (51), it is seen that

$$\left|\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \left[\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}}\cos\frac{\theta}{2}+\frac{24\eta(n,\theta)}{2n+3}\,\frac{\sin\left((n+1)\theta-\frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}\right]d\theta\right|$$

$$> \frac{\sqrt{\pi}}{16}\cdot\frac{1}{\sqrt{n+1}}\left(\sqrt{4(\mu+1)+1}-3\right)-\frac{12\pi}{\sqrt{n+1}}\left(1-\frac{1}{\sqrt{4\mu+1}}\right)$$

$$> c_1\sqrt{\frac{\mu}{n+1}}$$

for n and μ sufficiently large, c_1, as well as $c_2, c_3, \cdots$ which will be introduced below, signifying a certain positive constant, independent of n and μ. By (26) we have, for n sufficiently large,

$$\frac{2}{\pi}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} > c_2\sqrt{n+1},$$

so that we finally obtain from (48)

(52) $\quad |r_n\{F_n(0)\}| > c_3\sqrt{4\mu+5}$ for $n \geqq n'$, $\mu \geqq \mu'$, $2\mu \leqq n$.

On account of (27) there exists a c_4 such that

(53) $$\rho_n < c_4\sqrt{n+1} \qquad (n=1,2,\cdots).$$

We now select an infinite sequence of positive integers $n_1, n_2, n_3, \cdots$ subject to the conditions

$$n_1 \geqq n',$$

(54)
$$\omega\left(\frac{\pi}{n_i+1}\right) < \frac{c_3}{4c_4}\cdot\frac{1}{\sqrt{n_{i-2}+1}}\cdot\omega\left(\frac{\pi}{n_{i-1}+1}\right),$$

$$\Omega\left(\frac{1}{n_i}\right) < \frac{1}{\sqrt{n_{i-1}+1}}\,\omega\left(\frac{1}{n}\right) \qquad (i=3,4,5,\cdots).$$

The determination of such a sequence is possible, because after determining $n_1, n_2, \cdots, n_{i-1}$, we may satisfy the last two inequalities (54) by making n_i large enough, on account of $\lim_{\gamma=0} \omega(\gamma) = 0$ and $\lim_{\gamma=0} \Omega(\gamma)/\omega(\gamma) = 0$.

To each n_i we now associate a μ_i by the condition

$$\frac{4\mu_i + 1}{n_i + 1} < \frac{1}{n_{i-1} + 1} \leqq \frac{4\mu_i + 5}{n_i + 1}; \tag{55}$$

then any two intervals (μ_i, n_i) and (μ_j, n_j), in which, by (47), $F_{n_i}(\theta)$ and $F_{n_j}(\theta)$ respectively are not identically zero, have no points in common. Finally, we make v_i equal to zero or unity, according as

$$\left| r_{n_i} \left\{ \sum_{j=1}^{i-1} \frac{1}{2\pi} v_j \, \omega\left(\frac{\pi}{n_j + 1}\right) F_{n_j}(0) \right\} \right|$$

exceeds

$$\frac{c_3}{4\pi} \sqrt{\frac{n_i + 1}{n_{i-1} + 1}} \cdot \omega\left(\frac{\pi}{n_i + 1}\right)$$

or not. Then the expression

$$F(\theta) = \sum_{i=1}^{\infty} \frac{1}{2\pi} v_i \, \omega\left(\frac{\pi}{n_i + 1}\right) F_{n_i}(\theta) \tag{56}$$

has the properties required by theorem III.

Since for any given value of θ not more than one of the terms in the series (56) is different from zero (the intervals in which any two different functions $F_{n_i}(\theta)$ are not identically zero, having no point in common), the series is convergent. Furthermore, $F(\theta)$ satisfies the condition (28). To prove this, we first assume that θ and θ' belong to the same interval (μ_i, n_i); in this interval we have

$$F(\theta) = \frac{v_i}{2\pi} \omega\left(\frac{\pi}{n_i + 1}\right) \sin\left((n_i + 1)\theta - \frac{\pi}{4}\right).$$

We now determine an integer λ by the condition

$$\theta + \frac{2\lambda\pi}{n_i + 1} \leqq \theta' < \theta + \frac{2\lambda + 2}{n_i + 1}\pi$$

and denote by θ'' that one of the two quantities

$$\theta + \frac{2\lambda\pi}{n_i + 1} \quad \text{and} \quad \theta + \frac{2\lambda + 2}{n_i + 1}\pi$$

which is nearer to θ'; then we have

$$\left| \theta'' - \theta' \right| \leqq \frac{\pi}{n_i + 1}, \tag{57}$$

$$F(\theta') - F(\theta) = F(\theta') - F(\theta'')$$
$$= \frac{v_i}{2\pi}\omega\left(\frac{\pi}{n_i+1}\right)\cdot(n_i+1)(\theta'-\theta'')\cos\left((n_i+1)\xi - \frac{\pi}{4}\right),$$

ξ lying between θ' and θ'', and consequently

$$|F(\theta') - F(\theta)| < \frac{1}{2}\cdot\frac{\omega\left(\frac{\pi}{n_i+1}\right)}{\frac{\pi}{n_i+1}}|\theta''-\theta'|$$
$$= \frac{1}{2}\cdot\frac{\omega\left(\frac{\pi}{n_i+1}\right)}{\frac{\pi}{n_i+1}}\cdot\frac{|\theta''-\theta'|}{\omega(|\theta''-\theta'|)}\cdot\omega(|\theta''-\theta'|);$$

or, on account of (57) and the fact that $\omega(\gamma)/\gamma$ does not increase with γ,

$$(58) \qquad |F(\theta') - F(\theta)| < \tfrac{1}{2}\omega(|\theta''-\theta'|) \leqq \tfrac{1}{2}\omega(|\theta'-\theta|).$$

Next suppose that θ belongs to the interval (μ_i, n_i) and θ' to the interval (μ_j, n_j), and let θ_i be the end point of (μ_i, n_i) nearest to θ', and θ_j the end point of (μ_j, n_j) nearest to θ_i; then, since $F(\theta_i) = F(\theta_j) = 0$, we have

$$|F(\theta') - F(\theta)| = |F(\theta') - F(\theta_j) + F(\theta_j) - F(\theta_i) + F(\theta_i) - F(\theta)|$$
$$\leqq |F(\theta') - F(\theta_j)| + |F(\theta_i) - F(\theta)|.$$

Applying (58) and considering that $\omega(\gamma)$ does not decrease as γ increases, we find

$$|F(\theta') - F(\theta_j)| < \tfrac{1}{2}\omega(|\theta'-\theta_j|) \leqq \tfrac{1}{2}\omega(|\theta'-\theta|),$$
$$|F(\theta_i) - F(\theta)| < \tfrac{1}{2}\omega(|\theta_i-\theta|) \leqq \tfrac{1}{2}\omega(|\theta'-\theta|),$$

so that finally, for any values of θ and θ' in the interval $(0, \pi)$

$$|F(\theta') - F(\theta)| < \omega(|\theta'-\theta|).$$

Now $|\theta'-\theta| \leqq \gamma$, since $|\theta'-\theta|$ is one of the sides in the right-angled spherical triangle of which γ is the hypothenuse, and consequently

$$|F(\theta') - F(\theta)| < \omega(\gamma).$$

We now complete the proof by showing that

$$|\tau_{n_i}\{F(0)\}| \geqq K''\Omega\left(\frac{1}{n_i}\right)\sqrt{n_i} \qquad (i = 1, 2, 3, \cdots).$$

On account of (52) and (55), we have

$$\left|\tau_{n_i}\left\{\frac{1}{2\pi}\omega\left(\frac{\pi}{n_i+1}\right)F_{n_i}(0)\right\}\right| > \frac{c_3}{2\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right),$$

so that in case $v_i = 1$ we have, in view of the definition of v_i,

$$\left| r_{n_i}\left\{\sum_{j=1}^{i}\frac{1}{2\pi}v_j\,\omega\left(\frac{\pi}{n_j+1}\right)F_{n_j}(0)\right\}\right| \geqq \left| r_{n_i}\left\{\frac{1}{2\pi}\omega\left(\frac{\pi}{n_i+1}\right)F_{n_i}(0)\right\}\right|$$
$$-\left| r_{n_i}\left\{\sum_{j=1}^{i-1}\frac{1}{2\pi}v_j\,\omega\left(\frac{\pi}{n_j+1}\right)F_{n_j}(0)\right\}\right| > \frac{c_3}{2\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right)$$
$$-\frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right) = \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_{i+1}+1}\right).$$

In case $v_i = 0$, the definition of v_i gives

$$\left| r_{n_i}\left\{\sum_{j=1}^{i}\frac{1}{2\pi}v_j\,\omega\left(\frac{\pi}{n_j+1}\right)F_{n_j}(0)\right\}\right| = \left| r_{n_i}\left\{\sum_{j=1}^{i-1}\frac{1}{2\pi}v_j\omega\left(\frac{\pi}{n_j+1}\right)F_{n}(0)\right\}\right|$$
$$> \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right),$$

so that we always have

$$\left| r_{n_i}\left\{\sum_{j=1}^{i}\frac{1}{2\pi}v_j\,\omega\left(\frac{\pi}{n_j+1}\right)F_{n_j}(0)\right\}\right| > \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right). \tag{59}$$

From (56) and (59) it follows that

$$\left| r_{n_i}\left\{F(\theta)\right\}\right| \geqq \left| r_{n_i}\left\{\sum_{j=1}^{i}\frac{1}{2\pi}v_j\,\omega\left(\frac{\pi}{n_j+1}\right)F_{n_j}(\theta)\right\}\right|$$
$$-\left| r_{n_i}\left\{\sum_{j=i+1}^{\infty}\frac{1}{2\pi}v_j\,\omega\left(\frac{\pi}{n_j+1}\right)F_{n_j}(\theta)\right\}\right|, \tag{60}$$

and since

$$\left|\sum_{j=i+1}^{\infty}\frac{1}{2\pi}v_j\,\omega\left(\frac{\pi}{n_j+1}\right)F_{n_j}(\theta)\right| \leqq \frac{1}{2\pi}\omega\left(\frac{\pi}{n_{i+1}+1}\right)$$

for all values of θ, it follows from the definition of ρ_n and the inequality (53) that

$$\left| r_{n_i}\left\{\sum_{j=i+1}^{\infty}\frac{1}{2\pi}v_j\,\omega\left(\frac{\pi}{n_j+1}\right)F_{n_j}(\theta)\right\}\right|$$
$$\leqq \frac{1}{2\pi}\omega\left(\frac{\pi}{n_{i+1}+1}\right)\cdot\frac{1}{4\pi}\int_S |s_{n_i}(\cos\gamma)|\,d\sigma' = \frac{1}{2\pi}\omega\left(\frac{\pi}{n_{i+1}+1}\right)\cdot\rho_{n_i} \tag{61}$$
$$< \frac{c_4}{2\pi}\omega\left(\frac{\pi}{n_{i+1}+1}\right)\sqrt{n_i+1}.$$

Making $\theta = 0$, we conclude from (60), (59) and (61) that

$$|r_{n_i}\{F(0)\}| > \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right) - \frac{c_4}{2\pi}\omega\left(\frac{\pi}{n_{i+1}+1}\right)\sqrt{n_i+1}.$$

The second inequality in (54) shows that the right-hand expression is greater than

$$\frac{c_3}{8\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right) \geqq \frac{c_3}{8\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{1}{n_i}\right),$$

since

$$\frac{\pi}{n_i+1} \geqq \frac{1}{n_i},$$

and from the last inequality in (54) we finally obtain

$$\left|r_{n_i}\left\{F(0)\right\}\right| > \frac{c_3}{8\pi}\,\Omega\left(\frac{1}{n_i}\right)\sqrt{n_i+1} \qquad (i=1,2,3,\cdots)$$

which completes the proof of our theorem.

In the case of the Legendre series, Jackson has shown (l. c., I, theorem III) that there exists a K'' independent of n and a function $f(x)=f(\cos\theta)$ satisfying (5) such that

$$|\,r_n\{f(0)\}\,| > K''\,\omega\left(\frac{1}{n}\right)\log n$$

for an infinity of values of n. Apart from the introduction of $\sqrt{n}$ instead of $\log n$ necessitated by the asymptotic expression for ρ_n, our theorem III differs from Jackson's result in two respects: first, by the introduction of the function $\Omega(\gamma)$ converging toward zero with γ more strongly than $\omega(\gamma)$, and second, by the fact that while the two conditions

$$|f(\cos\theta')-f(\cos\theta)| \leqq \omega(|\cos\theta'-\cos\theta|) \qquad \text{(Jackson)},$$

$$|f(\cos\theta')-f(\cos\theta)| \leqq \omega(|\theta'-\theta|) \qquad \text{(present paper)}$$

are essentially equivalent in the interior of $0<\theta<\pi$, this is no longer the case in the vicinity of $\theta=0$, since

$$\lim_{\theta,\,\theta'=0}\frac{\cos\theta'-\cos\theta}{\theta'-\theta}=0.$$

CHICAGO, ILL.,
February 10, 1913.

LINEAR ASSOCIATIVE ALGEBRAS AND ABELIAN EQUATIONS*

BY

L. E. DICKSON

1. In the recent development of the theory of linear associative algebras the coördinates of whose numbers range over the field of ordinary complex numbers, an important rôle is played by the so-called simple algebras, the only ones being the matric algebras with r^2 units. For the more general case of algebras the coördinates of whose numbers range over any field (Körper) K of finite or infinite order, a corresponding importance is to be attached to simple algebras; an algebra is simple † if and only if it is the direct product of a matric algebra and a primitive algebra (one in which every element has an inverse). The problem of primitive algebras is the chief outstanding problem in the theory of algebras over a field K. Aside from real quaternions the only known primitive linear associative algebras are fields. We proceed to exhibit a primitive algebra L with r^2 units over K.

Let $\kappa(x) = 0$ be a uniserial abelian equation of degree r with respect to the field K, i. e., let the equation have as its coefficients numbers in K, be irreducible in K, and have as its roots

$$(1)\quad i,\ \theta(i),\ \theta^2(i)\equiv\theta[\theta(i)],\ \cdots,\ \theta^{s+1}(i)\equiv\theta^s[\theta(i)],\ \cdots,\ \theta^{r-1}(i),$$

where $\theta(x)$ is a polynomial with coefficients in K, while

$$(2)\qquad \theta^r(i) = i.$$

Then the algebra in question is the linear associative algebra L over K (i. e., the coördinates of its numbers are arbitrary elements of K) with the r^2 units

$$(3)\qquad i^s j^k \qquad (s, k = 0, 1, \cdots, r-1),$$

where

$$(4)\qquad ji = \theta(i)\, j,$$

$$(5)\qquad j^r = g \qquad (g \text{ in } K).$$

* Presented to the Society, April 14, 1906. Cf. the abstract in the Bulletin of the American Mathematical Society, vol. 12 (1905–6), p. 442. But §§ 9–22 were added March, 1913.

† J. H. M. Wedderburn, Proceedings of the London Mathematical Society, ser. 2, vol. 6 (1907), p. 99.

Using the associative law, we readily deduce (7) and hence

$$(6) \qquad l(i)\,j^s \cdot f(i)\,j^k = l(i)\,f[\theta^s(i)] \begin{cases} \cdot\, j^{s+k} & (\text{if } s+k<r) \\ \cdot\, g j^{s+k-r} & (\text{if } s+k \geqq r), \end{cases}$$

where $l(i)$ and $f(i)$ are any polynomials in i of degree $< r$ with coefficients in K. Conversely, it is readily verified that the associative law holds when multiplication is defined by the preceding relation and the supplementary law that the product of two polynomials in i is found as in ordinary algebra with a subsequent reduction of the degree to $r-1$ by use of the equation $\kappa(i)=0$ of degree r. We may of course write i_s for i^s and j_k for j^k in our formulæ. Hence to any uniserial abelian equation* of degree r with respect to K there corresponds a linear associative algebra L over K with r^2 units.

For $r=2$, L is a generalization of quaternions discussed in § 9; we may take $i^2=c$; then† L is primitive if and only if g is not the norm x^2-cy^2 of any number $x+yi$ in the field $K(i)$.

As stated in the abstract of 1906 (see the first reference above), an algebra L with r^2 units is primitive when g and the coefficients of $\kappa(x)$ are suitably chosen numbers of the field K. The argument in my manuscript of that date is as follows: Take $r=3$ and denote the general number of the algebra by $M=R+Sj+Tj^2$, where R, S, T are numbers of $K(i)$. For $M\neq 0$, the equation $XM=M'$ uniquely determines a number X of the algebra if and only if

$$\Delta \equiv \begin{vmatrix} R & gT_1 & gS_2 \\ S & R_1 & gT_2 \\ T & S_1 & R_2 \end{vmatrix}$$

vanishes only when $R=S=T=0$, where $R_1=R(\theta)$, $R_2=R(\theta^2)$,

$$S_1=S(\theta),\ S_2=S[\theta^2(i)], \qquad T_1=T(\theta),\ T_2=T[\theta^2(i)].$$

If $T=0$, $S\neq 0$, then $\Delta=0$ implies that $g=n(-R/S)$, where the norm SS_1S_2 of S is denoted by $n(S)$. If $T\neq 0$, T has an inverse in $K(i)$, so that it suffices to consider numbers M with $T=1$. Then

$$\Delta = g^2+g(SS_1S_2-RS_1-R_1S_2-R_2S)+RR_1R_2.$$

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If we set $h=g=1$, we lose the irreducibility property, but obtain the algebra ($i=e_1$, $j=e_1$) considered by Wedderburn, Proceedings of the Royal Society of Edinburgh, vol. 26 (1906), pt. 1, p. 48.

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If $S = 0$, then $\Delta = 0$ implies that $g = 0$ or $n(-g/R)$, according as $R = 0$, or $R \neq 0$. Henceforth, let $S \neq 0$. To simplify Δ, set

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Then Δ has the factor $n^2(S)$ and $\Delta = 0$ implies that

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The product of this by $G - 1$ may be written in the form

$$Gn(Q - 1) = n(Q - G).$$

Thus, if $Q \neq 1$, G (and hence g) is the norm of a number of $K(i)$. If $Q=1$, then $(G - 1)^2 = 0$. Hence *if g is not the norm of any number of $K(i)$, the algebra L is primitive.*

For a general value of r the last theorem was found in November, 1913, by Professor Wedderburn; his brief and elegant proof will appear in an early number of these Transactions.

It will be proved that any linear associative algebra having the properties specified in § 2 contains a subalgebra of type L. It is later proved that there exist such algebras, not identical with their subalgebras L. Some remarkable algebras are obtained in this way.

2. Let A be any linear associative algebra the coördinates of whose numbers range over any given field F and for which the following three properties hold:

(a) There exists in A a number i satisfying an equation $\phi(x) = 0$ of degree n with coefficients in F and irreducible in F.

(b) Any number of A which is commutative with i is in the field $F(i)$.

(c) There exists in A a number j, not in $F(i)$, such that $ji = \theta j + \sigma$, where θ and σ are in $F(i)$.

Note that the field $F(i)$ is composed of the totality of polynomials in i with coefficients in F. Indeed, in view of the associative law, any two powers of i are commutative; hence any two polynomials in i are commutative. That the quotient of two polynomials in i can be expressed as a polynomial in i is proved as in the theory of algebraic numbers.*

Conditions (a)–(c) hold when A is the system of quaternions with coördinates in the field of real numbers. Conditions (a) and (b) are satisfied for any linear associative algebra D over F in which each right-hand and left-hand division is always possible and unique. In fact, any number i of D satisfies an equation with coefficients in F and irreducible in F. Choose i so that the degree n of the equation is a maximum. If a number v of D is

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commutative with i, it is in the field $F(i)$. For if not, a standard theorem on algebraic numbers shows that the field $F(i, v)$ would contain a number satisfying an equation of degree exceeding n and irreducible in F.

For a linear associative algebra over F satisfying condition (a), the number of units is a multiple of n; they may be taken to be

$$i^s,\ i^s j,\ i^s j_1,\ \cdots \qquad (s = 0, 1, \cdots, n-1).$$

In case the number of units is $2n$, ji is necessarily a linear combination of the i^s, $i^s j$ and hence is of the form $\theta j + \sigma$ in (c).

These remarks serve to indicate the origin of conditions (a)–(c). All three conditions are satisfied by any algebra of type L (§ 8).

3. From the relation in (c), we find by induction that

$$ji^s = \theta^s j + (\theta^{s-1} + \theta^{s-2} i + \cdots + \theta i^{s-2} + i^{s-1})\,\sigma,$$

for $s > 1$. We are here using ordinary powers of θ and not symbolical powers as in (1). Multiply the members of the equation by c_s and sum, where

$$\phi(x) = \sum_{s=0}^{n} c_s x^s$$

is the function in (a) with the root i. Thus

$$0 = \phi(\theta) j + c_1 \sigma + \sum_{s=2}^{n} (\theta^{s-1} + \theta^{s-2} i + \cdots + \theta i^{s-2} + i^{s-1})\, c_s \sigma.$$

But j is not in the field $F(i)$. Hence $\phi(\theta) = 0$. Suppose first that $\theta = i$. If $\sigma = 0$, then $ji = ij$, j not in $F(i)$, contrary to (b). Hence $\sigma \neq 0$. The preceding relation thus gives

$$0 = c_1 + \sum_{s=2}^{n} s i^{s-1} c_s = \phi'(i).$$

Thus i would be a double root of $\phi(x) = 0$, contrary to its irreducibility. Hence $\theta \neq i$. Thus $Ji = \theta J$ if

$$J = j + \frac{\sigma}{(\theta - i)}.$$

Hence after a suitable choice of j we have relation (4) and

$$ji^s = [\theta(i)]^s j \qquad (s = 1, \cdots, n).$$

If $f(x)$ is any polynomial with coefficients in F,

$$(7) \qquad jf(i) = f(\theta) j, \quad \cdots, \quad j^s f(i) = f[\theta^s(i)]\, j^s,$$

the last relation following by induction and the use of the notation defined in (1). In particular, if $f(x)$ is the function $\phi(x)$ which vanishes for $x = i$,

we see from the first relation that $\theta(i)$ is a root of $\phi(x) = 0$. Hence $\phi[\theta(x)] = 0$ has the root i in common with the irreducible equation $\phi(x)=0$ and therefore the root $\theta(i)$. Thus $\theta^2(i)$ is a root of $\phi(x) = 0$. We see in this way that each of the numbers (1) is a root of $\phi(x) = 0$. Hence there must be an equality

$$\theta^{k+r}(i) = \theta^k(i) \qquad (r > 0).$$

If $k = 0$, (2) holds. If $k > 0$, $x = \theta(i)$ is a root of

$$\theta^{k+r-1}(x) = \theta^{k-1}(x). \tag{8}$$

Since this equation with coefficients in the field F has a root in common with the equation $\phi(x) = 0$, irreducible in F, it has also the root $x = i$. Then (8) for $x = i$ gives a relation like the initial equality but with lower symbolic powers. Proceeding in this manner, we ultimately obtain relation (2).

Let r be the least positive integer for which (2) holds. If $r = 1$, (4) would be in contradiction with (b). The argument just used shows that the numbers (1) are all distinct.

Theorem 1. *The algebra A contains a number j not in the field $F(i)$ and satisfying relation* (4). *Among the roots of $\phi(x) = 0$ occur the r ($r > 1$) distinct numbers* (1); *they form a closed cycle since relation* (2) *holds.*

4. In the last equation (7) take $s = r, f(x) = x$. By (2),

$$j^r i = \theta^r(i)\, j^r = ij^r.$$

Hence, by (b), j^r is in the field $F(i)$:

$$j^r = g(i), \tag{9}$$

where $g(i)$ is a polynomial with coefficients in F. Henceforth we shall assume that $g(i) \neq 0$.

For $\rho = 1$ and $\rho = 2$ we know that the numbers

$$i^s,\ i^s j,\ i^s j^2,\ \cdots,\ i^s j^{\rho-1} \quad (s = 0, 1, \cdots, n-1;\ 0 < \rho < r) \tag{10}$$

are linearly independent with respect to the field F. Assume that the ρn numbers (10) are linearly independent. Then these and the n numbers $i^s j^\rho$ are linearly independent. For, if not, there is a polynomial $\tau(i) \neq 0$ such that τj^ρ is a linear function of the numbers (10) with coefficients in F. Since τ has an inverse in the field $F(i)$,

$$j^\rho = R_0 + R_1 j + \cdots + R_{\rho-1} j^{\rho-1}, \tag{11}$$

where each R_k is in $F(i)$. Multiply each term of (11) by i on the right and apply (7). We get

$$\theta^\rho(i)\, j^\rho = R_0 i + R_1 \theta(i)\, j + \cdots + R_{\rho-1} \theta^{\rho-1}(i)\, j^{\rho-1}.$$

Eliminating j^ρ by means of (11) and noting that the numbers (10) are linearly independent with respect to F, we see that

$$\theta^\rho(i)\,R_k = R_k\,\theta^k(i) \quad [k = 0, 1, \cdots, \rho - 1;\ \theta^0(i) \equiv i].$$

Since $k < \rho < r$, $\theta^\rho(i) \neq \theta^k(i)$ by Theorem I. Hence each $R_k = 0$. Then $j^\rho = 0$ by (11), so that $j^r = 0$, contrary to the assumption on (9).

Theorem 2. *The algebra A contains a linear associative subalgebra S over F with the rn units*

$$i^s,\quad i^s j,\quad i^s j^2,\quad \cdots,\quad i^s j^{r-1} \quad (s = 0, 1, \cdots, n-1) \tag{12}$$

linearly independent with respect to the field F. The product of any two of these units can be expressed as a linear function of the units by means of $\phi(i) = 0$ and relations (4) *and* (9).

5. In view of the first relation (7),

$$\begin{gathered} ji = \theta(i)\,j, \qquad j\theta(i) = \theta^2(i)\,j, \\ j\theta^2(i) = \theta^3(i)\,j, \quad \cdots, \qquad j\theta^{r-1}(i) = ij, \end{gathered} \tag{13}$$

where the final equation has been simplified by use of (2). Hence if s_k is an elementary symmetric function of the numbers (1),

$$js_k = s_k\,j. \tag{14}$$

Let K be the field composed of all rational functions with coefficients in F of these elementary symmetric functions:

$$K = F(s_1, s_2, \cdots, s_r). \tag{15}$$

Hence any number of K is commutative with j and, being a polynomial in i, also with i. Thus any number of K is commutative with every number of the algebra S.

If a number N of S is commutative with every number of S, then N is in the field K. For, by (*b*), N is in the field $F(i)$. Next, if $0 \leqq s < r$, $Nj^s = j^s N$ requires, by (7), that

$$N(i)\,j^s = N[\theta^s(i)]\,j^s.$$

Multiply each member by j^{r-s} on the right. Since the terms of (9) are not zero, we get

$$N(i) = N[\theta^s(i)] \qquad (s = 0, \cdots, r-1).$$

If we exclude the case of fields F having a modulus dividing r, we obtain, by addition and division by r, $N(i)$ expressed as a symmetric function of the numbers (1) and hence as a number of K. But we can easily give a proof valid for all fields F. By means of the equation of degree r having the roots

(1), we can express $N(i)$ as a polynomial in i of degree $r-1$ with coefficients in K. Thus

$$N(x) = a_0 + a_1 x + \cdots + a_{r-1} x^{r-1} \qquad (a\text{'s in } K)$$

holds for $x = i$. But the left member is unaltered when x is replaced by any one of the r numbers (1). Since an equation of degree $r-1$ with r distinct roots is an identity, $N(i) = a_0, a_1 = 0, \cdots$.

Theorem 3. *A number of the subalgebra S over F with the units* (12) *is commutative with every number of S if and only if it is in the field K defined by* (15).

Corollary. The right member of (9) is in the field K.

6. Let $\psi(x) = 0$ be the equation with coefficients in K and irreducible in K which has the root i. Since j is commutative with every number of K, in view of (14), relations (7) hold for any polynomial $f(x)$ with coefficients in K. Taking $\psi(x)$ as $f(x)$, we conclude that every $\theta^s(i)$ is a root of $\psi(x) = 0$. Thus the latter has the r distinct roots (1). But these r numbers are the roots of an equation $\kappa(x) = 0$ with coefficients in K, in view of the definition of K. Hence $\kappa(x)$ is irreducible in K.

Theorem 4. *The r numbers* (1) *are the roots of an equation $\kappa(x) = 0$ of degree r with coefficients in the field K and irreducible in K.*

7. Any number of algebra S, i. e., any linear function of its units (12) with coefficients in F, can be expressed by means of the equation $\kappa(i) = 0$ of degree r as a linear function of the r^2 numbers (3) with coefficients in the field K. These r^2 numbers are linearly independent with respect to K; the proof is as in § 4 with n replaced by r, F by K, $F(i)$ by $K(i) \equiv F(i)$. Hence the algebra S over F can be exhibited as the algebra L over K defined in § 1.

Theorem 5. *Any linear associative algebra A over the field F with the properties (a), (b), (c), and such that $g(i) \neq 0$ in* (9), *has a subalgebra S over F which can be exhibited as the algebra L over K with r^2 units* (3) *linearly independent with respect to K.*

8. If N is a number of the algebra L which is commutative with i, then N is a polynomial in i with coefficients in the field K. Indeed,

$$N = R_0 + R_1 j + \cdots + R_{r-1} j^{r-1},$$

where each R_s is a polynomial in i with coefficients in K. By (7),

$$j^s i = \theta^s(i)\, j^s,$$

$$Ni = R_0 i + R_1 \theta(i)\, j + \cdots + R_{r-1} \theta^{r-1}(i)\, j^{r-1}.$$

Equating the latter to iN and noting that the units (3) are linearly independent with respect to K, and that the numbers (1) are distinct, we find that $R_1, R_2, \cdots, R_{r-1}$ are zero.

Theorem 6. *Any algebra of type L has the properties* (*a*), (*b*), (*c*), *with n replaced by r and F by K.*

Algebras A With r = 2.

9. For $r = 2$, the irreducible equation satisfied by i may be given the form* $i^2 = c$ by applying a linear transformation on i. The algebra L then has the units $1, i, j, k$, where

$$\begin{gathered} i^2 = c, \quad j^2 = g, \quad ij = k = -ji, \\ k^2 = -cg, \quad jk = -gi = -kj, \quad ki = -cj = -ik. \end{gathered} \tag{16}$$

The relations in the second line were derived from those in the first line by use of the associative law. Right-hand and left-hand division† is always possible and unique in this algebra L if and only if c is not the square of a number in K and g is not expressible in the form $x^2 - cy^2$, with x and y in K. We shall assume henceforth that these conditions are satisfied. Then L is a generalized‡ quaternion algebra over K.

Consider an algebra A related to this L as in Theorem 5. Then K is of rank $n/2$ with respect to its subfield F. When K is so related to F, algebra L can evidently be exhibited as an algebra S of $2n$ units over F. We seek algebras A with such a subalgebra S but distinct from S. Since right- and left-hand division is always possible in S, the number of units m of A is a multiple of $2n$.

10. For the simplest case $n = 2$, $K \equiv F$ and A has L as a subalgebra. Let A have the least possible number 8 of units:

$$1, \quad i, \quad j, \quad k, \quad f, \quad if, \quad jf, \quad kf.$$

Then $fi = Q + Rf$, where Q and R are in L. Multiply by i on the right. Thus $R^2 = c$, $Qi + RQ = 0$. Hence $R = \pm i$. If $R = i$, then $Q = ej + lk$, and i is commutative with $f + jl/2 + ke/(2c)$, contrary to (*b*). Hence $R = -i$, $Qi = iQ$, $Q = e + li$. Taking $f - l/2 - ie/(2c)$ as a new unit f, we have

$$fi = -if. \tag{17}$$

Next, $fj = v + wf$, where v and w are in L. Multiply by j on the right. Thus $w^2 = g$, $w = \pm j$, $vj \pm jv = 0$.

* We exclude fields having modulus 2.

† These Transactions, vol. 13 (1912), p. 66. Division is always uniquely possible in the more general algebra (9), p. 67, since $\Delta(X) = \Delta'(X) = \sigma^2$.

‡ The ordinary quaternion algebra if $c = g = -1$ and if K is the field of all real numbers.

For the upper signs, $v = ri + sk$. Taking $f - kr/(2g)$ as a new f, we see that (17) remains true and that

$$fj = sk + jf. \tag{18}$$

Multiply (17) on the right by j. Thus $fk = -scj - kf$. Multiply (17) on the left by f. Thus f^2 is commutative with i. Hence, by (b),

$$f^2 = a + bi \qquad (a, b \text{ in } F). \tag{19}$$

Multiply (18) on the left by f. Thus $bk = -s^2 cj - bk$, $b = s = 0$,

$$fi = -if, \quad fj = jf, \quad fk = -kf, \quad f^2 = a. \tag{20}$$

The general element is $Q + Rf$, Q and R in L. We have

$$(Q + Rf)(q + rf) = Qq + Rr^* a + (Qr + Rq^*) f, \tag{20'}$$

where r^* is derived from r by changing the signs of the coefficients of i and k. Since $fr = r^* f$, $(r\rho)^* = r^* \rho^*$, the algebra with the multiplication table (20′) is associative. It is not an algebra A over F since every element is commutative with jf. Indeed, $Qj = jQ^*$.

For the lower signs, $v = e + lj$, so that

$$fj = e + lj - jf.$$

Multiply (17) on the right by j. We get

$$fk = -ei - lk + kf. \tag{21}$$

Now the multiplication table (16) for L is unaltered in form if we set

$$j = k_1, \quad k = cj_1, \quad g = -cg_1.$$

Then, by (21),

$$fj_1 = -\frac{e}{c} i - lj_1 + j_1 f.$$

Dropping the subscripts, we have the former case $fj = v + jf$.

Theorem 7. *There does not exist an algebra of type A with 8 units.*

Nor is there any algebra $A > L$ in which every number satisfies a quadratic equation with coefficients in F. For, if so, we may set $f^2 =$ constant. Then $(f \pm i)^2$ is a linear function of $f \pm i$, so that $fi + if$ is a linear function of $f + i$ and also of $f - i$ and hence is a constant:

$$fi + if = c_1, \quad fj + jf = c_2, \quad fk + kf = c_3,$$

$$fk = fij = (-if + c_1) j = i(jf - c_2) + c_1 j = kf - c_2 i + c_1 j.$$

Hence $2fk$ and thus f is in L. Thus* $A \equiv L$.

* We may thus give a remarkably simple proof of the famous theorem that a linear associative algebra (over the field of reals) in which division is always possible and unique is either the field of reals, the field of ordinary complex numbers or the system of real quaternions. Indeed, it is very easily shown that if there be more than two units, the algebra has a subalgebra of the quaternion type [making use of these Transactions, vol. 13 (1912), p. 64, first lines of § 6]. See § 11 of *Linear Algebras* (in press), Cambridge Tracts.

Algebras A With 16 Units.

11. For the next case $n = 4$, there exist algebras A with the minimum number 16 of units. We may take K to be the field $F(\xi)$, where

$$\xi^2 = \nu \qquad (\nu \text{ not a square in } F). \tag{22}$$

As the 8 units of algebra S we may take

$$1,\ \xi,\ i,\ \xi i = i\xi,\ j,\ \xi j = j\xi,\ k,\ \xi k = k\xi. \tag{23}$$

The products of these by f on the right may be taken as the remaining 8 units of algebra A. Then $f\xi = Q + Rf$, where Q and R are in S. Multiplying by ξ on the right, we see that

$$R^2 = \nu \equiv \xi^2, \qquad (\xi + R)\, Q = 0.$$

If $\xi + R \neq 0$, then $Q = 0$, $R = \xi$, $f\xi = \xi f$, so that A is an algebra of 8 units over the field $F(\xi)$ and hence, if existent, is a quaternary algebra L over a larger field (§ 10). We therefore take $R = -\xi$. Taking $f - Q/(2\xi)$ as a new f, we have

$$f\xi = -\xi f. \tag{24}$$

Thus ξ is commutative with $q + rf$, where q and r are in S, if and only if $r = 0$. But ξ is commutative with f^2. Hence

$$f^2 = \lambda + \mu j \qquad [\lambda, \mu \text{ in } F(i)]. \tag{25}$$

12. The following notations will be employed. If $u = a + b\xi$, where a and b are in the field F, write $u' = a - b\xi$. If $\alpha = r + si$, where r and s are in the field $F(\xi)$, write $\bar{\alpha} = r - si$. Hence

$$fu = u'f, \qquad j\alpha = \bar{\alpha}j, \qquad k\alpha = \bar{\alpha}k \quad [u \text{ in } F(\xi),\ \alpha \text{ in } F(i)]. \tag{26}$$

13. We must have

$$fi = Q + Rf, \qquad Q = \alpha + \beta j, \qquad R = \gamma + \delta j \quad [\alpha, \cdots, \delta \text{ in } F(i)].$$

Then

$$fi^2 = c'f = Qi + RQ + R^2 f, \qquad c' = R^2, \qquad Qi + RQ = 0,$$

$$c' = \gamma^2 + \delta\bar{\delta}g, \qquad \delta(\gamma + \bar{\gamma}) = 0, \tag{27}$$

$$g\bar{\beta}\bar{\delta} = -\alpha(i + \gamma), \qquad \bar{\alpha}\delta = \beta(i - \gamma). \tag{28}$$

First, let $\delta = 0$. If either α or β is not zero, then

$$\gamma = \pm i, \qquad c' = \gamma^2 = c,$$

and i satisfies the equation $i^2 = c$, with c in F, contrary to $n = 4$. Hence $\alpha = \beta = 0, fi = \gamma f$. Now $\gamma = x + yi$, x, y, and c' being in $F(\xi)$. Thus $c' = \gamma^2$ gives $xy = 0$. If $y = 0$, then $i^2 = c = x'^2$, and $i = \pm x'$ would be n $F(\xi)$. Hence $x = 0$,

$$(\mathrm{I}) \qquad fi = yif, \qquad c' = y^2 c \qquad [y \text{ in } F(\xi)].$$

If a number of A is commutative with i it is easily seen (see § 19) to be in $F(i)$, so that property (b) holds.

Next, let $\delta \neq 0$. Then by (27_2), $\gamma = \sigma i$, σ in $F(\xi)$. If $\beta \neq 0$, we find by eliminating α from (28), and using (27_1), that $c' = c$. Hence $\beta = 0$, $\alpha = 0$, and

$$(\mathrm{I}') \qquad fi = (\sigma i + \delta j) f, \qquad c' = c\sigma^2 + g\delta\bar{\delta} \quad [\sigma \text{ in } F(\xi),\ \delta \neq 0 \text{ in } F(i)].$$

14. We must have $fj = q + rf$, q and r in S. Then

$$(29) \qquad fj^2 = g'f = qj + rq + r^2 f, \qquad g' = r^2, \qquad qj = -rq.$$

Set $q = B + Cj$, $r = D + Ej$, where $B, \cdots, E$ are in $F(i)$. Then

$$(30) \qquad g' = D^2 + gE\bar{E}, \qquad E(D + \bar{D}) = 0,$$

$$(31) \qquad gC + DB + gE\bar{C} = 0, \qquad B + DC + E\bar{B} = 0.$$

First, let $E = 0$. Set $D = w + zi$, w and z in $F(\xi)$. Since $D^2 = g'$ is in $F(\xi)$, $wz = 0$. If $z = 0$, then $j^2 = g = w'^2$, and $j = \pm w'$ would be in $F(\xi)$. Hence $z \neq 0$, $D = zi$. Eliminating B from (31), we get $C(g - D^2) = 0$. If $C \neq 0$, then $j = \pm zi$. Hence $C = 0$, $B = 0$, and

$$(\mathrm{II}) \qquad fj = zif, \qquad g' = z^2 c \qquad [z \text{ in } F(\xi)].$$

Next, let $E \neq 0$. Then, by (30), $D = \omega i$, ω in $F(\xi)$, and

$$(32) \qquad g' = c\omega^2 + gE\bar{E}.$$

Multiply (31_1) by i and substitute the resulting value of $c\omega B$ into the product of (31_2) by ωc. By use of (32), we get

$$iC(g' - g) = 0.$$

If $g' \neq g$, then $C = 0$ and, by (31), $B = 0$, since the case $E\bar{E} = 1$, $\omega = 0$, is excluded by (32). Hence

$$(\mathrm{II}') \qquad fj = (\omega i + Ej) f, \qquad g' \neq g, \qquad E \neq 0 \text{ [subject to (32)]}.$$

For $g' = g$, it is simpler to employ conditions (29). Then $r^2 = j^2$, $r = \pm j$.

If $r = j$, set $f_1 = if$. Then (I) and (I′) hold with f replaced by f_1 and δ by $-\delta$. Also

$$f_1 j = i(q + rf) = iq - jif = iq - jf_1.$$

Hence it suffices to treat the case $r = -j$. Then $fj + jf = q$ is commutative with ξ. But, by (24), $(fj + jf)\xi = -\xi(fj + jf)$. Hence

$$(II'') \qquad fj = -jf, \qquad g' = g.$$

15. It remains to combine one of the cases (I), (I′) with one of the cases (II), (II′), (II″). The combination (I) and (II) is readily excluded. Multiply (I) on the right by j and apply (II). We get

$$fk = yi \cdot zif = yzcf = fy'z'c', \qquad k = y'z'c',$$

whereas k is not in $F(\xi)$.

16. Let (I) and (II″) hold. By the latter, $f^2 j = jf^2$. Hence by (25) and (26_2), $\lambda = \bar{\lambda}$, $\mu = \bar{\mu}$, so that λ and μ are in $F(\xi)$. By (I),

$$f^2 i = yy'if^2, \qquad y^2 y'^2 = 1.$$

Using (25), we see that $\mu = 0$ or $\lambda = 0$, according as $yy' = +1$ or -1. For the present, let the first alternative hold:

$$(33) \qquad yy' = +1, \qquad f^2 = \lambda,$$

where λ is in $F(\xi)$. Then f is commutative with λ. Hence by (26_1), λ is in F.

Writing* f_1, f_2, f_3 for if, jf, kf, we may express any number of the algebra as a linear combination of $1, i, j, k, f, f_1, f_2, f_3$, with coefficients in the field $F(\xi)$. The associative law then uniquely determines the multiplication table for these eight units:

	i	j	k	f	f_1	f_2	f_3
i	c	k	cj	f_1	cf	f_3	cf_2
j	$-k$	g	$-gi$	f_2	$-f_3$	gf	$-gf_1$
k	$-cj$	gi	$-gc$	f_3	$-cf_2$	gf_1	$-gcf$
f	yf_1	$-f_2$	$-yf_3$	λ	$y\lambda i$	$-\lambda j$	$-y\lambda k$
f_1	ycf	$-f_3$	$-ycf_2$	λi	$c\lambda y$	$-\lambda k$	$-yc\lambda j$
f_2	$-yf_3$	$-gf$	gyf_1	λj	$-y\lambda k$	$-g\lambda$	$gy\lambda i$
f_3	$-ycf_2$	$-gf_1$	$ygcf$	λk	$-yc\lambda j$	$-g\lambda i$	$gyc\lambda$

* The rest of this section and § 17 are not essential in the later work.

Let N be unity or one of the seven numbers $i, \cdots, f_3$ at the head of the table. Let N_1 be one of the same eight numbers. Let x and u be any numbers in $F(\xi)$. Then

$$(34) \qquad (xN)(uN_1) = (xU)(NN_1) \quad \begin{pmatrix} U = u \text{ if } N = 1, i, j, k; \\ U = u' \text{ if } N = f, f_1, f_2, f_3 \end{pmatrix}.$$

By means of the table and (34), we can express the product of any two linear functions of $1, i, \cdots, f_3$ with coefficients in $F(\xi)$ as a third such function. Multiplication thus uniquely defined is associative; it is far simpler to prove this fact by means of the condensed notations of § 18.

17. The preceding algebra can be given a simpler form by using a functional notation in addition to the notations in § 12. If $A = u + vi$, where u and v are in $F(\xi)$, write $A^* = u' + vv'i$. Then by (26), (I), (II″),

$$(35) \qquad fA = A^* f, \quad jA = \bar{A} j, \quad \phi A = \bar{A}^* \phi,$$

where $\phi = jf$. Any number of the algebra is of the form

$$N = X + Yj + Zf + W\phi \quad [X, \cdots, W \text{ in } F(i)].$$

Then

$$(36) \qquad N(A + Bj + Cf + D\phi) = P + Qj + Rf + S\phi,$$

where

$$(37) \qquad \begin{aligned} P &= XA + Y\bar{B}g + ZC^*\lambda - W\bar{D}^*\lambda g, \\ Q &= XB + Y\bar{A} - ZD^*\lambda + W\bar{C}^*\lambda, \\ R &= XC + Y\bar{D}g + ZA^* - W\bar{B}^* g, \\ S &= XD + Y\bar{C} - ZB^* + W\bar{A}^*. \end{aligned}$$

18. A further simplification is effected by writing the general number of the algebra in the form $U + Vf$, where U and V are in S (i. e., are generalized quaternions), on writing $U\dagger = X^* - Y^* j$ if $U = X + Yj$, X and Y being in $F(i)$. Then $fU = U\dagger f$. The law of multiplication is now

$$(38) \qquad (U + Vf)(G + Hf) = UG + VH\dagger\lambda + (UH + VG\dagger)f.$$

This multiplication is seen at once to be associative since the element $f^2 = \lambda$ of F is commutative with every U and since

$$(39) \qquad (UV)\dagger = U\dagger V\dagger, \quad (U\dagger)\dagger = U.$$

The latter relations follow at once from

$$(40) \quad (X+Y)^* = X^* + Y^*, \quad (XY)^* = X^* Y^*, \quad (X^*)^* = X, \quad (\bar{X})^* = \bar{X}^*,$$

where X, Y are any numbers in $F(i)$. Relations (40) are easily verified, use being made of (I), (33), $(u')' = u$, $(uv)' = u'v'$.

19. Let i be commutative with $U + Vf$. Then $Ui = iU$, so that U is in $F(i)$, and $yVi = iV$. Multiply the latter by i on the left. Then

$$cV = i^2 V = yiVi = y(yVi)i = y^2 cV.$$

If $y^2 = 1$, then $c' = c$ by (I) and i is a root of the equation $i^2 = c$ with coefficients in F, contrary to $n = 4$. Hence $V = 0$. Thus any number of the algebra commutative with i is in the field $F(i)$.

20. Since $yy' = 1$, the condition in (I) may be written $c'y' = cy$. Hence cy is in the field F, so that $c = dy'$, d in F. Set $y = a - b\xi$, a and b being in F. Then $yy' = 1$ gives

$$a^2 - \nu b^2 = 1. \tag{41}$$

Now i is a root of $x^2 = c$ and hence of

$$x^2 = d(a + b\xi), \qquad (x^2 - da)^2 = d^2 b^2 \nu.$$

By using (41), we see that i is a root of

$$x^4 - 2dax^2 + d^2 = 0. \tag{42}$$

Theorem 8. *If a, b, d, λ, g, ν are numbers in the field F such that* (41) *holds and such that equations* (42) *and $\xi^2 = \nu$ are irreducible in F, and if we set $y = a - b\xi$, $c = dy'$, the linear algebra over F with the* 16 *units* $1, i, j, k, f, f_1, f_2, f_3$ *and their products by ξ on the left, with multiplication defined by* (34) *and the table in* § 16, *or more compactly by* (38), *is a linear associative algebra with the properties* (a), (b), (c) *of* § 2.

The conditions on a, b, d are easily discussed. For example, let F be the field of all rational numbers. Take $\nu = -1$, $d = \pm 1$, $a = 0$, $b = 1$. Then (41) holds and (42) becomes the equation $x^4 + 1 = 0$ for the primitive eighth roots of unity and is known to be irreducible in the field of rational numbers.

Then $y = -\xi$, $c = \pm\xi$, and K is the field of the numbers $r + s\xi$, where r and s are rational and $\xi^2 = -1$. While it is not an essential condition on our algebra, we may choose g so that division* is unique in the algebra L.

21. We have treated the first one (33) of two alternatives. Next, let

$$yy' = -1, \qquad f^2 = \mu j.$$

* Evidently c is not the square of $r + s\xi$, since then $2s^2 = +1$ or -1. Next, there is a rational number g not of the form $x^2 - cy^2$, where $x = r + s\xi$, $y = l + m\xi$. Indeed, $x^2 - \xi y^2 = A + B\xi$, $A = r^2 - s^2 + 2lm$, $B = 2rs - l^2 + m^2$. Take $B = 0$. If $r = 0$, then $A = \pm 2l^2 - s^2$. If $r \neq 0$, we eliminate s and get $4r^2 A = \{2r^2 + (l+m)^2\}\{2r^2 - (l-m)^2\}$. If $2r^2 - v^2$ is divisible by the prime p, but not by p^2, then $p = 8k \pm 1$. For $2r^2 + v^2$ the condition is $p = 8k + 1$ or $8k + 3$. The primes not in either set are $p = 8k + 5$. Hence we may take as g a number divisible by an odd power of a prime $8k + 5$.

By $ff^2 = f^2 f$, we get $\mu' = -\mu$. Thus $\mu = t\xi$,

$$yy' = -1, \qquad f^2 = t\xi j \qquad (t \text{ in } F). \tag{43}$$

Define A^* as in § 17, and $U\dagger$ as in § 18. As before,

$$fA = A^* f, \qquad fU = U\dagger f, \qquad (AB)^* = A^* B^*, \qquad (UV)\dagger = U\dagger V\dagger, \tag{44}$$

but we now have

$$(A^*)^* = \bar{A}, \qquad (U\dagger)\dagger j = jU, \qquad [(X + Yj)\dagger]\dagger = \bar{X} + \bar{Y}j, \tag{45}$$

of which the last two equations are equivalent. Multiplication

$$(U + Vf)(G + Hf) = UG + VH\dagger t\xi j + (UH + VG\dagger) f \tag{46}$$

is easily verified to be associative. Also § 19 holds here. By (I),

$$c' y' \xi' = cy\xi, \qquad c = dy'\xi \qquad (d \text{ in } F)$$

As in § 20, $a^2 - \nu b^2 = -1$, while i is a root of

$$x^4 - 2bd\nu x^2 + \nu d^2 = 0. \tag{47}$$

The first condition is satisfied if $a = b = 1$, $\nu = 2$, and for $d = 1$ equation (47) becomes $x^4 - 4x^2 + 2 = 0$, which is irreducible in the field of rational numbers. *Hence we have an algebra for which* (*a*), (*b*), (*c*) *hold.*

Theorem 9. *If* a, b, d, t, g, ν *are numbers in the field* F *such that* $a^2 - \nu b^2 = -1$ *and such that* (47) *and* $\xi^2 = \nu$ *are irreducible in* F, *if* $y = a - b\xi$, $c = dy'\xi$, *and if* U *and* V *are numbers of the algebra* (16) *over* $F(\xi)$, *then the numbers* $U + Vf$, *subject to the law of multiplication* (46), *define a linear associative algebra with* 16 *units and having the properties* (*a*), (*b*), (*c*) *of* § 2.

22. Finally, let (I) and (II′) hold. Multiply (I) by j on the right and replace the fj introduced by its value in (II′). Thus

$$fk = y\omega c + yEkf.$$

Multiply this by k and eliminate the fk introduced. Then

$$-c' g' f = y^2 (Ek)^2 f + y\omega c (1 + yE) k.$$

If $\omega \neq 0$, then $yE = -1$ and, by the coefficients of f, $cg = c' g'$. Now y and hence E is in $F(\xi)$, so that $\bar{E} = E$. Hence $y^2 E\bar{E} = 1$. Multiply this by c and apply (I). Thus $c' E\bar{E} = c$. But (32) then gives $\omega = 0$.

Hence $\omega = 0$. Consider the case $yE = -1$. By (I), $yy' = \pm 1$. Thus $E = \mp y'$, $g' = gy'^2$. Let $J = k$, $K = cj$. Then

$$fi = yif, \quad fJ = -Jf, \quad iJ = K, \quad J^2 = G, \quad K^2 = -cG,$$

where $G = -cg = G'$. Hence the algebra is equivalent to that in § 16.

As it is not our purpose to enumerate all the algebras A with 16 units, we do not consider the remaining values of E, nor an algebra in which (I′) holds.

Theorems 8 and 9 show that there exist linear associative algebras A with the properties (a), (b), (c), and distinct from their sub-algebras S. Hence the class of algebras A is more extensive than the class of primitive algebras L.

SOME THEOREMS CONCERNING GROUPS WHOSE ORDERS ARE POWERS OF A PRIME*

BY

WILLIAM BENJAMIN FITE

Professor Burnside has called attention to certain limitations on the nature of the derived groups of groups of prime power order.† One of these limitations is that such a derived group cannot be non-abelian and have a cyclic central. Theorem I of the present paper gives a slight generalization of this limitation, and an obvious modification of the argument used in the proof of this theorem establishes the important result that the second central of a group whose order is a power of an odd prime cannot be cyclic.

The results given at the end of the paper are continuations of some obtained by Professor Burnside in the article just cited.

Throughout the paper frequent use is made of the theorem that in a group whose order is a power of a prime the operations which correspond to the invariant operations of the ith cogredient are commutative with all of the ith commutators of the group.‡

We shall represent by G a group of class k and order p^m, where p is a prime; by G' the first cogredient of G; by H_i the ith central§ of G; by K_j the jth commutator subgroup; and by $K_{j,i}$ the subgroup of K_j that is contained in H_i.

Suppose that $K_{j,2}$ is cyclic and generated by the operation t of order p^e. If $k > 3$, every operation of $K_{j,3}$ of order p must be contained in $K_{j,2}$ since the operations of this latter group are invariant in K_j and its operations of order p are invariant in G. Hence $K_{j,3}$ has only one subgroup of order p. It follows at once that $K_{j,3}$ is cyclic if p is odd. If $p = 2$, $K_{j,2}$ contains an operation of order 2^2 that is invariant in $K_{j,3}$. But in the non-cyclic group of order a power of 2 with only one subgroup of order 2 no operation of

* Presented to the Society at the Cleveland meeting, January 1, 1913.

† Proceedings of the London Mathematical Society, ser. 2, vol. 11 (1912), pp. 241, 242.

‡ Fite, these Transactions, vol. 7 (1906), p. 62.

§ I follow here the notation of Burnside's *Theory of Groups*, first edition, p. 62, rather than that of the second edition, p. 120. Cf. de Séguier, *Éléments de la théorie des groupes abstraits*, p. 87.

order 2^2 is invariant. We conclude therefore that in this case also $K_{j,3}$ is cyclic.

Suppose then that $K_{j,3}$ is generated by the operation t_1 of order $p^{\alpha+\beta}$. If we denote by $K'_{j,i}$ the subgroup of the jth commutator subgroup of G' that is contained in the $(i-1)$th central of G', it follows from what has been proved that $K'_{j,4}$ must be cyclic. Hence the quotient group $K_{j,4}/K_{j,3}$ must be cyclic. If it is of order p^γ and t_2 is an operation of $K_{j,4}$ that corresponds to a generator of it, $t_2^{p^\gamma}$ is a power of t_1 whose exponent is relatively prime to p, and therefore $K_{j,4}$ is cyclic. An obvious continuation of this argument shows that K_j is cyclic. This establishes the following

Theorem I. *The jth commutator subgroup of a group of prime power order is cyclic if those of its operations that are contained in the second central of the group form a cyclic subgroup.*

Since $K_{1,2}$ is contained in the central of K_1, Theorem I includes as a special case Professor Burnside's theorem referred to in the introduction.

In case p is odd an argument closely analogous to the one used in the proof of Theorem I shows that if H_2 were cyclic G itself would be cyclic. But if G were cyclic there would be no H_2. This proves

Theorem II. *The second central of a group of order p^m, where p is an odd prime, cannot be cyclic.*

If $p=2$, the preceding argument does not apply, since the operations of H_2 are not all necessarily invariant in H_3 and therefore an operation of H_3 of order 2 may give a commutator of order 2^2. We can however proceed as follows.

If $k>3$, H_3 contains a commutator t_1 that is not in H_2. Now $\{H_2, t_1\}$ is an abelian group with only one subgroup of order 2, since t_1 is commutative with every one of its commutators. Hence this group is cyclic. If H_3 contained an operation s of order 2 that is not in $\{H_2, t_1\}$, s would transform t_1 into itself multiplied by an invariant commutator of G of order 2. Hence s would be commutative with t, and accordingly every operation of G would transform it into itself multiplied by an invariant commutator of G. But this is not consistent with the fact that s is not in H_2. We conclude therefore that H_3 contains only one subgroup of order 2. It contains the operation t_1 and every one of its operations transforms t_1 into itself multiplied by an operation of H_1. But this would be impossible in case H_3 were non-cyclic. We conclude therefore that H_3 is cyclic.

The continuation of the argument up to the point of showing that H_{k-1} must be cyclic is the same as in the case of an odd prime. But beyond this point the argument breaks down, since G/H_{k-1} has no commutator besides the identity. As a matter of fact, there are groups of order 2^m whose second centrals are cyclic—for example, the diedral groups of these orders when $m>3$.

Suppose now that K_1 is cyclic and generated by the commutator t of order

p^λ, where p is odd. If $k > 3$, G/H_{k-2} is metabelian* with a cyclic commutator subgroup. If p^r is the order of this subgroup, G/H_{k-3} must contain an invariant commutator of order p^r and the invariant commutators of G/H_{k-3} must form a cyclic subgroup of order p^s, where $s \geqq r$.

Since any operation of G/H_{k-3} that corresponds to an invariant operation of G/H_{k-2} is commutative with every commutator of G/H_{k-3}, no operation of H_{k-1} can transform t into itself multiplied by a commutator that is not in H_{k-3}. But some operation of G must transform t into $t^{1+xp^{s_1}}$, where x is relatively prime to p, and $r \leqq s_1 < r+s$, since otherwise t would be contained in H_{k-2} and G/H_{k-2} would be abelian. But this is impossible since G is of class k.

Suppose then that A is so selected that

$$A^{-1}tA = t^{1+xp^{s_1}},$$

where x is relatively prime to p, and s_1 is a minimum. Then

$$A^{-1}t^{p^{r-1}}A = t^{p^{r-1}+xp^{r+s_1-1}}, \qquad A^{-1}t^{p^r}A = t^{p^r+xp^{r+s_1}}.$$

Now since t^{p^r} is contained in H_{k-2}, while $t^{p^{r-1}}$ is not, it follows that

$$r + s_1 \geqq r + s, \qquad r + s_1 - 1 < r + s.$$

Hence $s_1 = s$, and therefore

$$A^{-p^r}tA^{p^r} = t^{(1+xp^s)^{p^r}}.$$

But

$$(1 + xp^s)^{p^r} \equiv 1 + xp^{r+s} \pmod{p^{2s+1}}.$$

Moreover $r + s < 2s + 1 \leqq \lambda$, since the lowest power of t that is contained in H_{k-3} is the p^{r+s}th power and since H_{k-3} must contain a commutator of order p^s. Hence

$$(1 + xp^s)^{p^r} \not\equiv 1 \pmod{p^\lambda},$$

and A^{p^r} is not commutative with t. Since however G/H_{k-2} is metabelian and contains no commutator of order greater than p^r, A^{p^r} must be contained in H_{k-1}. We conclude therefore that *if K_1 is cyclic and $k > 3$, not all of the operations of K_1 are invariant in H_{k-1}.*

This argument with a slight modification is valid when $p = 2$, provided that $s > 1$; but when $s = 1$ it breaks down and the conclusion does not hold, as may be seen from the groups of order 2^5.

We have assumed that $k > 3$. The existence of metabelian groups with cyclic commutator subgroups shows that the conclusion does not hold when

* I use the term *metabelian* in the sense defined by me in these Transactions, vol. 3 (1902), p. 331. Burnside uses the term in a different sense. See his *Theory of Groups*, second edition, p. 57.

$k = 2$. When $k = 3$ every operation of H_2 is commutative with t. But the following conclusion from this argument for $k > 3$ does hold when $k = 3$; namely,* *if K_1 is cyclic, H_{k-1} cannot coincide with K_1.*

The commutator formed by any two operations, t_1 and t_2, of K_1 is contained in H_{k-3}, since G/H_{k-3} is of class three and the commutator subgroup of a group of the third class is abelian. The index of $K_{1,\,k-3}$ under K_1 is at least p^2. If it is exactly p^2, we can assume that t_2 is equal to the product of some operation of H_{k-2} and t_1. Hence† the commutator formed by t_1 and t_2 is in H_{k-4} and the index of the commutator subgroup of K_1 under K_1 is at least p^3.

If t_1 and t_2 are operations of K_i they are ith commutators or products of ith commutators and correspond to invariant operations of the ith cogredient of G/H_{k-2i-1}, since they are contained in H_{k-i}. Hence the commutator formed by t_1 and t_2 is contained in H_{k-2i-1}. Now if K_i is non-abelian the index of $K_{i,\,k-2i-1}$ under K_i is at least p^{i+1}. If it is exactly p^{i+1}, we can show by an argument similar to the one used in the special case just considered that the commutator formed by t_1 and t_2 is contained in H_{k-2i-2}. Hence the index of the commutator subgroup of K_i under K_i is at least p^{i+2} if K_i is non-abelian.

If the jth derived group is contained in H_x, the $(j+1)$th derived group is contained in $H_{x-(j+1)}$, provided that $x > j+1$. Hence the ith derived group is contained in $H_{k-i(i+1)/2}$, and therefore in a group of order p^m whose ith derived group is not the identity we must have $k > i(i+1)/2$. Hence $m \geqq 2 + i(i+1)/2$. For $i > 4$ this exceeds the lower limit for m given by Professor Burnside.

COLUMBIA UNIVERSITY,
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* de Séguier, *loc. cit.*, p. 127. This follows also from Theorem II.

† This result is given by Burnside in the article cited in the introduction.

LIMITS IN TERMS OF ORDER, WITH EXAMPLE OF LIMITING ELEMENT NOT APPROACHABLE BY A SEQUENCE*

BY

RALPH E. ROOT

1. Introduction.

The importance of the rôle of the order relation in the theory of the linear continuum has been recognized since the appearance, in 1872, of Dedekind's *Stetigkeit und irrationale Zahlen*,† and it received added importance at the hands of Cantor‡ in 1895. A later contribution to the theory was made by O. Veblen§ in 1905, and a systematic account of the theory of the continuum as a type of order was given by E. V. Huntington || the same year. The work of these writers is directed toward a complete characterization of the linear continuum in terms of order alone. A set or class of elements, otherwise undefined, is assumed to fulfil conditions, stated in terms of order, sufficiently restrictive to admit of effective use of the class in the rôle of range of the continuous independent variable.

Meanwhile certain classes have been recognized as being, in effect, the range of an independent variable, while not fulfilling all the conditions of the linear continuum. The desirability of utilizing analogies that exist between theories that pertain to these classes and the theory of the continuum is obvious. To this end it is important to generalize, as far as may be, the notions of point sets so as to be applicable to such classes.

The thesis of M. Fréchet¶ was a notable effort toward this form of generalization. He presupposes a definition of limit of a sequence of elements, the definition fulfilling certain mild conditions, and defines in terms of this the notion of limiting element of a subclass, and develops a theory applicable to any class for which a limit relation can be defined fulfilling the prescribed

* Presented to the Society (Chicago), March 21, 1913.

† English translation by W. W. Beman, in a volume titled Dedekind's *Essays on the Theory of Numbers*, 1901.

‡ Mathematische Annalen, vol. 46 (1895), p. 481.

§ These Transactions, vol. 6 (1905), p. 165.

|| Annals of Mathematics, ser. 2, vol. 6 (1905), p. 151, and vol. 7 (1905), p. 15.

¶ Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), p. 6.

conditions. He later adopts an analogue of the distance function (*voisinage*), the restrictions of which give a much more extensive theory.

E. H. Moore, in his paper at the Rome Congress in 1908,* and in his later published work on general analysis,† introduces a form of generalization of different character, considering properties of classes whose elements are real valued functions of a variable absolutely unconditioned. This general variable admits of a wide range of special applications.

E. R. Hedrick‡ contributes to the theory of such general classes by defining, in terms of the limit relation used by Fréchet, an "enclosable property" which replaces the voisinage or distance function used by the latter to define a special type of limit. In a paper by T. H. Hildebrandt§ in which the *voisinage* of Fréchet is replaced by a K relation of a type used by E. H. Moore, it is shown that most of the theorems proved by Fréchet may be based on milder assumptions than he used. The papers by Fréchet, Hedrick and Hildebrandt each consists of two stages, the first stage pertaining to a range possessing a mildly conditioned definition of limit of a sequence, the second stage pertaining to a range conditioned in such manner as to involve a notion closely analogous to that of "neighborhood," and introducing in some form the notion of "uniformity on the range."

In a former paper‖ by the present writer postulates are stated directly in terms of "neighborhood," but are free from any requirement of uniformity on the range. This absence of uniformity facilitates the treatment of ideal limiting elements, while invariance of postulates under the process of composition of classes is utilized to simplify the treatment of multiple and iterated limits. The paper may, in some sense, be regarded as an intermediate stage between the two stages of each of the three papers just mentioned.

The object of the present paper is to use in this field of generalization some of the simple properties of ordered classes. A triadic relation of the type of "betweenness" serves in the definition of segment, which, with certain mild postulates that are fulfilled by any simply ordered class, is effective in the development of a theory of limits and continuity. The postulates possess the invariance under composition and the freedom from uniformity that are

* *On a Form of General Analysis, with Application to Linear Differential and Integral Equations*, Atti del IV° Congresso Internazionale dei Matematici, Roma, 1908, vol. 2, p. 98.

† *Introduction to a Form of General Analysis*, published in a volume titled *The New Haven Mathematical Colloquium*, 1910.

‡ *On Properties of a Domain for which any Derived Set is Closed*, these Transactions, vol. 12 (1911), pp. 285–294.

§ *A Contribution to the Foundations of Fréchet's Calcul Fonctionnel*, American Journal of Mathematics, vol. 34 (1912), p. 237.

‖ *Iterated Limits in General Analysis*, American Journal of Mathematics, Jan. and Apr., 1914.

emphasized in the previous paper, and avoid some of the restrictions of that paper.

The theory developed in sections 2–7 may be characterized as a " neighborhood theory," being based on a definition of limiting element in terms of segment, a type of neighborhood. In section 8 we indicate the character of a " sequential theory "* based on the same postulates, which emerges by the adoption of a definition of limiting element in terms of " limit of a sequence," as is done in the first stage of the paper by Fréchet. This form of definition is used also by Hedrick, and in the first part of the paper by Hildebrandt. Finally, in section 9, we define a " sequential property " for classes of elements, the presence of which, in addition to Postulates I–III, is a necessary and sufficient condition for the equivalence of the two theories considered. It is shown that simply ordered classes exist which do not possess this property.† A fourth postulate, sufficient to secure the sequential property, is discussed in relation to certain fundamental assumptions that have occurred in discussions of the linear continuum and of general analysis.

2. Definitions in a system $(\mathfrak{Q}; B)$.

The system $(\mathfrak{Q}; B)$ with which we are concerned consists of a class $\mathfrak{Q}$ of elements q of any kind whatever, and a relation B defined for triads of elements of $\mathfrak{Q}$. The relation B is said to be defined for a class $\mathfrak{Q}$ if for every three elements q_1, q_2, and q_3 (distinct or not) the question " does the relation $B_{q_1 q_2 q_3}$ hold " has a definite answer, yes or no. A simple instance of such a system is obtained if we take for $\mathfrak{Q}$ the class of all points on a line, and let B denote the ordinary relation of " betweenness," so that $B_{q_1 q_2 q_3}$ may be read " q_2 is between q_1 and q_3."

The following definitions are found convenient in developing a theory pertaining to a system $(\mathfrak{Q}; B)$.

Definition 1. If q_1 and q_2 are such that $B_{q_1 q q_2}$ holds for some q, the class of all elements q such that $B_{q_1 q q_2}$ holds is called a *segment*.

Such a segment is denoted by $\mathfrak{S}_{q_1 q_2}$ or, in case clearness permits, simply by $\mathfrak{S}$.

* T. H. Hildebrandt (*loc. cit.*, p. 239) uses the term "sequential continuity" with respect to functions on a range where limiting element has a definition of this form.

† In the Bulletin of the American Mathematical Society, vol. 17, (1911), p. 538, we gave a brief forecast of the principal features of the paper cited above on iterated limits. The closing sentence of that forecast states that the treatment is available for any class having linear order. The statement is exact, but the application then in view would give a theory for linear sets substantially equivalent to the "sequential" theory for such sets indicated in the present paper. This result is attained by assigning to each element not approachable by a sequence a neighborhood consisting of the element itself, thus making it an isolated element also in the neighborhood theory. In the last section of the present paper we indicate a restriction on linear sets sufficient to render the neighborhood theory of this paper a special case under the general theory of the paper cited above.

DEFINITION 2. An element q is a *limiting element of a subclass* $\mathfrak{P}$ of $\mathfrak{Q}$ if every segment containing q contains an element p of $\mathfrak{P}$ distinct from q.

DEFINITION 3. A subclass $\mathfrak{P}$ is *closed* if every limiting element of $\mathfrak{P}$ is in $\mathfrak{P}$.

DEFINITION 4. A subclass $\mathfrak{P}$ is *dense in itself* if every element of $\mathfrak{P}$ is a limiting element of $\mathfrak{P}$.

DEFINITION 5. A subclass $\mathfrak{P}$ is *compact* if every infinite subclass of $\mathfrak{P}$ has a limiting element.*

DEFINITION 6. An element q is *interior* to a subclass $\mathfrak{P}$ if there is a segment containing q and contained in $\mathfrak{P}$.

DEFINITION 7. An element q is a *limit of a sequence* $\{q_n\}$ if for every segment $\mathfrak{S}$ containing q there is an $n_{\mathfrak{S}}$ such that, for $n > n_{\mathfrak{S}}$, q_n is in $\mathfrak{S}$.

DEFINITION 8. The *derived class* of a class $\mathfrak{P}$ is the class of all limiting elements of $\mathfrak{P}$.

DEFINITION 9. The *frontier class* of a class $\mathfrak{P}$ is the class of all elements in $\mathfrak{P}$ or the derived class of $\mathfrak{P}$ that are not interior to $\mathfrak{P}$.

3. THEOREMS REQUIRING NO RESTRICTIONS ON THE SYSTEM.

The theorems of the present section hold for any system of the type $(\mathfrak{Q}; B)$ in the absence of any restrictions whatever.† The proofs, being entirely obvious, are omitted.

THEOREM I. *If* $\mathfrak{L}$ *is the derived class of* $\mathfrak{P}$, *then the class* $\mathfrak{P} + \mathfrak{L}$ *is closed.*

THEOREM II. *The frontier class of any class is closed.*

THEOREM III. *If q is a limiting element of a subclass* $\mathfrak{P}$ *it is a limiting element of every class containing* $\mathfrak{P}$.‡

THEOREM IV. *A class* $\mathfrak{P}$ *is compact if and only if every subclass of* $\mathfrak{P}$ *is compact.*§

THEOREM V. *A class formed of a finite number of compact classes is compact.*

THEOREM VI. *A sequence formed by repeating a single element has that element as limit.*

THEOREM VII. *If a sequence has a limit every infinite subsequence has the same limit.*

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* Fréchet, loc. cit., p. 6.

† It is common, under similar circumstances, to state theorems as corollaries of definitions. We desire to emphasize the fact that the office of a definition is to conserve economy in the use of language, and that it is, in reality, a part of the statement of every proposition which involves the term it defines. The validity of a theorem depends on the definitions of its terms only in the same sense that it depends on its own hypothesis.

‡ A proposition of this form is used by F. Riesz as a postulate in his paper on *Stetigkeitsbegriff und abstracte Mengenlehre*, Atti del IV° Congresso Internazionale dei Matematici, Roma, 1908, vol. 2, pp. 18–24. Propositions of the form of his other postulates occur as theorems in section 7.

§ This theorem and the next are given by Fréchet, *loc. cit.*, p. 7. Theorems VI and VII are similar in form to postulates used by Fréchet. Further reference to his work is made in section 7.

It should be understood that the limit of a sequence need not be unique. We are able to prove this property only after the adoption of Postulate III.

THEOREM VIII. *If q is interior to $\mathfrak{P}$, all but a finite number of terms of any sequence whose limit is q are interior to $\mathfrak{P}$.**

THEOREM IX. *If there exists a sequence of distinct elements of $\mathfrak{P}$ having q as limit, q is a limiting element of $\mathfrak{P}$.*

We note that the converse of this proposition holds only in a very restricted system. It is because of this that we find a divergence of the "neighborhood" and the "sequential" theories. We discuss this question more fully in the last two sections.

4. THEOREMS IN A COMPOSITE SYSTEM.

From two systems $(\mathfrak{Q}'; B')$ and $(\mathfrak{Q}''; B'')$ of the type under consideration we derive a system $(\mathfrak{Q}; B)$ of the same type. The class $\mathfrak{Q}$ is the product class $\mathfrak{Q}'\,\mathfrak{Q}''$, consisting of all elements q of the form $q = q'\,q''$, where q' is in $\mathfrak{Q}'$ and q'' is in $\mathfrak{Q}''$. The element q is not to be regarded as a product, but simply as the pair $q'\,q''$. For three elements q_1, q_2, q_3 the relation $B_{q_1 q_2 q_3}$ holds if and only if $q_1 = q_1'\,q_1''$, $q_2 = q_2'\,q_2''$ and $q_3 = q_3'\,q_3''$, and the relations $B'_{q_1' q_2' q_3'}$ and $B''_{q_1'' q_2'' q_2''}$ hold. Thus $(\mathfrak{Q}; B)$ is a definite system of the required type, and is called the composite system of the systems $(\mathfrak{Q}'; B')$ and $(\mathfrak{Q}''; B'')$. Similarly any number of systems of this type give rise to a definite composite system.†

A system $(\mathfrak{Q}; B)$ may fulfil the following postulate:

POSTULATE I. *If q is an element of $\mathfrak{Q}$ there exists a segment containing q.*

If the relation B is interpreted as ordinary betweenness, and fulfils conditions sufficient to render $\mathfrak{Q}$ a simply ordered class, then the addition of this postulate is equivalent to the assumption that there is no first or last element. This is not, in effect, any restriction; for any simply ordered class may be regarded as a subclass of a class having no extreme elements. Moreover, if we so modify the relation B that B_{qqq_1} and $B_{q_1 qq}$ shall hold in case q is an extreme element, while it remains otherwise as ordinary betweenness, then we find that Postulate I is no restriction on a simply ordered class.

If each of two or more systems fulfil Postulate I their composite system does also. And if a composite system fulfils this postulate so must every component system, providing that $\mathfrak{Q}$ in the composite system contains at least one element. Excluding the trivial case just provided against, we may say that Postulate I is invariant under composition of systems.

The theorems of this section are, for the sake of clearness and brevity, stated for a system $(\mathfrak{Q}; B)$, composite of two systems, $(\mathfrak{Q}'; B')$ and $(\mathfrak{Q}''; B'')$.

* E. R. Hedrick, loc. cit., p. 286.

† Compare T. H. Hildebrandt, loc. cit., p. 250.

The notations $\mathfrak{P}$, $\mathfrak{R}$, etc., indicate subclasses of $\mathfrak{Q}$, and with suitable accents they denote subclasses of $\mathfrak{Q}'$ and $\mathfrak{Q}''$. The manner of extending to the case of a composite system of several systems is entirely obvious.

THEOREM X. *A class $\mathfrak{P}$ is a segment if and only if there exist segments $\mathfrak{S}'$ and $\mathfrak{S}''$ such that $\mathfrak{P}$ is the product class $\mathfrak{S}'\,\mathfrak{S}''$.*

This proposition is obviously true, even for systems in which Postulate I is not assumed to hold.

THEOREM XI. *If $\mathfrak{P}$ is the product class $\mathfrak{P}'\,\mathfrak{P}''$ and if $q = q'\,q''$, then a necessary and sufficient condition that q shall be a limiting element of $\mathfrak{P}$ is that q' is a limiting element of $\mathfrak{P}'$ and q'' is in $\mathfrak{P}''$, or q'' is a limiting element of $\mathfrak{P}''$ and q' is in $\mathfrak{P}'$, or q' and q'' are limiting elements respectively of $\mathfrak{P}'$ and $\mathfrak{P}''$.*

That the condition is sufficient is easily seen, without the use of the postulate. The necessity of the condition may be shown to involve the postulate as follows: Let q' be in a segment containing no other element, and suppose q'' is not in any segment; then q is not in any segment and is therefore a limiting element of every $\mathfrak{P}$, but q' is not a limiting element of any $\mathfrak{P}'$. By considering $\mathfrak{P} = \mathfrak{P}'\,\mathfrak{P}''$, where $\mathfrak{P}'$ does not contain q', we see that the condition is not necessary. The proof of the necessity of the condition is very simple if Postulate I is fulfilled.

THEOREM XII. *If $\mathfrak{P} = \mathfrak{P}'\,\mathfrak{P}''$, then $\mathfrak{P}$ is closed if both $\mathfrak{P}'$ and $\mathfrak{P}''$ are closed, and, in case $\mathfrak{P}$ contains at least one element, both $\mathfrak{P}'$ and $\mathfrak{P}''$ are closed if $\mathfrak{P}$ is closed.*

The first part of this theorem involves the necessity of the condition in Theorem XI, and therefore the use of the postulate, but the second part of the theorem involves only the sufficiency of the condition, and can be proved for an unrestricted system.

THEOREM XIII. *If $\mathfrak{P} = \mathfrak{P}'\,\mathfrak{P}''$, then $\mathfrak{P}$ is dense in itself if and only if one of the component classes, $\mathfrak{P}'$ or $\mathfrak{P}''$, is dense in itself.*

The proof may be arranged, without use of the postulate, as follows: Suppose $\mathfrak{P}'$ is dense in itself. An element p is of the form $p'\,p''$, and a segment $\mathfrak{S}$ containing p is of the form $\mathfrak{S}'\,\mathfrak{S}''$, where $\mathfrak{S}'$ contains p' and $\mathfrak{S}''$ contains p''. $\mathfrak{S}'$ contains a p_1' distinct from p', and $p_1 = p_1'\,p''$ is in $\mathfrak{S}$ and distinct from p. Thus every p is seen to be a limiting element of $\mathfrak{P}$, and $\mathfrak{P}$ is dense in itself. Now suppose neither $\mathfrak{P}'$ nor $\mathfrak{P}''$ is dense in itself, then there exists p' in a segment $\mathfrak{S}'$ containing no other element of $\mathfrak{P}'$, and there exists p'' in a segment $\mathfrak{S}''$ containing no other element of $\mathfrak{P}''$, thus the element $p = p'\,p''$ is in the segment $\mathfrak{S} = \mathfrak{S}'\,\mathfrak{S}''$ which contains no other element of $\mathfrak{P}$. This element p is not a limiting element of $\mathfrak{P}$, so that $\mathfrak{P}$ is not dense in itself.

THEOREM XIV. (*a*) *If $\mathfrak{P} = \mathfrak{P}'\,\mathfrak{P}''$, then if $\mathfrak{P}$ is compact both $\mathfrak{P}'$ and $\mathfrak{P}''$ are compact, provided that each contains at least one element.*

(*b*) *If $\mathfrak{P} = \mathfrak{P}'\,\mathfrak{P}''$ and $\mathfrak{P}'$ and $\mathfrak{P}''$ are both compact, then every infinite product class in $\mathfrak{P}$ has a limiting element.*

The proof of (a) requires the use of Postulate I, but (b) holds for any system. It will be observed that the compactness of the two component classes does not imply the compactness of $\mathfrak{P}$, except in case of a more restricted system.*

THEOREM XV. *If $\mathfrak{P} = \mathfrak{P}'\,\mathfrak{P}''$ and $q = q'\,q''$, then q is interior to $\mathfrak{P}$ if and only if q' and q'' are interior to $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively.*

This theorem clearly does not involve Postulate I.

THEOREM XVI. *If $q = q'\,q''$, then a necessary and sufficient condition that q shall be the limit of the sequence $\{ q_n \}$ is that the sequences $\{ q_n' \}$ and $\{ q_n'' \}$, where $q_n = q_n'\,q_n''$ for every n, shall have respectively the limits q' and q''.*

As in Theorem XI, the necessity of the condition involves the postulate, while the sufficiency can be proved for an unrestricted system. The proof, being very simple, is omitted.

5. LIMITS AND CONTINUITY OF FUNCTIONS.

Functions defined on subclasses of $\mathfrak{Q}$ are denoted by the letters μ, φ, θ, etc. A function μ is said to be defined on a subclass $\mathfrak{P}$ if to every element p there corresponds a definite function value μ_p, which is a real number, or $+\infty$ or $-\infty$. With respect to a given function μ on a given class $\mathfrak{P}$ we consider, relative to a definite limiting element l of $\mathfrak{P}$, three symbols,

$$\lim_{p \to l} \mu_p, \quad \overline{\lim_{p \to l}}\, \mu_p, \quad \underline{\lim_{p \to l}}\, \mu_p,$$

which denote respectively the limit, the upper limit and the lower limit of μ as p approaches l. In this section 5, the symbol $p \to l$ under "lim" is omitted; it is to be understood in every case that we mean the limit as p approaches l.

DEFINITION 10. Following are the conditions under which the *limit*, the *upper limit* and the *lower limit* of μ on $\mathfrak{P}$ exist at a given limiting element l.

(a) $\lim \mu_p = a$ is equivalent to the condition: For every e there exists a segment $\mathfrak{S}_e$ containing l such that for every p in $\mathfrak{S}_e$ distinct from l we have $| \mu_p - a | \leqq e$.†

(b) $\overline{\lim}\, \mu_p = a$ is equivalent to the two conditions: (1) For every e there exists an $\mathfrak{S}_e$ containing l such that if p is in $\mathfrak{S}_e$ and distinct from l then $\mu_p \leqq a + e$. (2) For every $\mathfrak{S}$ containing l and every e there exists a p in $\mathfrak{S}$ distinct from l such that $\mu_p \geqq a - e$.

(c) $\overline{\lim}\, \mu_p = +\infty$ is equivalent to the condition: For every $\mathfrak{S}$ containing l and every a there is a p in $\mathfrak{S}$ distinct from l such that $\mu_p > a$.

* See Theorem VIII (d) of chapter II of the paper on *Iterated Limits in General Analysis*, loc. cit. In section 9 of the present paper conditions sufficient to validate such a theorem are stated for a system $(\mathfrak{Q}; B)$.

† The letter a denotes a real number, while e invariably denotes a positive real number.

(*d*) $\overline{\lim}\,\mu_p = -\infty$ and $\underline{\lim}\,\mu_p = -\infty$ are each equivalent to the condition: For every a there exists an $\mathfrak{S}_a$ containing l such that if p is in $\mathfrak{S}_a$ and distinct from l then $\mu_p < a$.

(*e*) Conditions for $\overline{\lim}\,\mu_p = a$, $\underline{\lim}\,\mu_p = -\infty$ are analogous to conditions in (*b*) and (*c*), while conditions for $\underline{\lim}\,\mu_p = +\infty$ and $\lim \mu_p = +\infty$ are analogous to (*d*).

THEOREM XVII. *If a function μ defined on $\mathfrak{P}$ is such that for a given limiting element l of $\mathfrak{P}$ there exists an a such that for every p distinct from l we have $|\mu_p| \leqq a$, then there exist uniquely $\bar{a}$ and $\underline{a}$ such that*

$$\overline{\lim_{p \to l}}\,\mu_p = \bar{a}, \quad \underline{\lim_{p \to l}}\,\mu_p = \underline{a}.$$

For each $\mathfrak{S}$ containing l let $\bar{a}_{\mathfrak{S}}$ be the least upper bound of μ_p for all elements p in $\mathfrak{S}$ distinct from l. It is not difficult to see that the greatest lower bound of $\bar{a}_{\mathfrak{S}}$ for all segments $\mathfrak{S}$ containing l is the $\bar{a}$ required. Similarly the existence of $\underline{a}$ may be shown. The proof depends on Postulate I.

If in the hypothesis of this theorem we remove the restriction that μ is bounded on the range obtained by omitting the element l from $\mathfrak{P}$, then the least upper bound of μ_p on a given $\mathfrak{S}$, excluding $p = l$, may be $+\infty$ or $-\infty$, so that the greatest lower bound of these least upper bounds may be $+\infty$ or $-\infty$. But in any case the latter quantity definitely exists, and is the upper limit of μ at l. Since a similar procedure gives the lower limit of μ at l, we have the following corollary.

COROLLARY. *For every μ defined on $\mathfrak{P}$ and for every limiting element l of $\mathfrak{P}$ the upper limit of μ at l and the lower limit of μ at l exist.*

But making use of Postulate I it is not possible to prove that $\overline{\lim}\,\mu_p \geqq \underline{\lim}\,\mu_p$, and the existence of a limit for μ at a limiting element l neither implies nor is implied by the equality of the upper and lower limits of μ at l. Furthermore, the limit of a function at a limiting element, when it exists, is not necessarily unique. These points may be established by the following examples.

EXAMPLE 1. Let Ω consist of the points (x, y) of a plane, and let $B_{q_1 q_2 q_3}$ hold if the three points are on a line and q_2 is between q_1 and q_3 in the usual sense. Postulate I is fulfilled. Let $\mathfrak{P}$ be the class of points in the first quadrant, and let μ be defined on $\mathfrak{P}$ so that for $p = (x, y)$ we have $\mu_p = y/x$, then for $l = (0, 0)$ we have $\overline{\lim}\,\mu_p = 0$ and $\underline{\lim}\,\mu_p = +\infty$, while for every positive a we have the conditions fulfilled for $\lim \mu_p = a$, and also we have $\lim \mu_p = +\infty$.

EXAMPLE 2. With the same system $(\Omega; B)$ and the same class $\mathfrak{P}$ as above, take μ on $\mathfrak{P}$ so that for $p = (x, y)$ we have μ_p equal to unity if $x = y$, equal to one half if x is rational and greater than y, equal to zero if x is irrational and greater than y, equal to 2 if x is rational and less than y, and equal to 3 if x

is irrational and less than y. We now have a unique limit, unity, at the origin, while the upper limit is one half and the lower limit is 2.

EXAMPLE 3. Again with the same system and the same class $\mathfrak{P}$, take μ on $\mathfrak{P}$ so that for $p = (x, y)$ we have μ_p equal to unity for x rational, equal to zero for x irrational and equal to or greater than y, and equal to 2 for x irrational and less than y. The upper limit and the lower limit at the origin have the same value, unity, while the limit does not exist.

We have, however, the theorem:

THEOREM XVIII. *If μ is defined on $\mathfrak{P}$, then μ can have no limit at a given limiting element l of $\mathfrak{P}$ which is greater than the lower limit of μ at l or less than the upper limit of μ at l.*

As an immediate corollary we have

COROLLARY. *If the upper and lower limits are equal, then if a limit exists it is unique and equal to the upper and lower limits.*

For further theorems we require a second postulate, which may conveniently take the following form:

POSTULATE II. *If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are segments containing q, then there is a common subclass of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ which is a segment containing q.*

This condition is clearly invariant under composition of systems, provided that the composite class contains at least one segment. Several theorems result from this postulate.

THEOREM XIX. *If μ is defined on $\mathfrak{P}$, then for every limiting element l of $\mathfrak{P}$ we have* $\underline{\lim}_{p \to l} \mu_p \leq \overline{\lim}_{p \to l} \mu_p$.

THEOREM XX. *For a given μ on $\mathfrak{P}$ and a given limiting element l of $\mathfrak{P}$, a necessary and sufficient condition for the existence of a limit of μ at l is the equality of the upper and lower limits of μ at l.*

COROLLARY. *If a limit of μ at l exists, it is unique and equal to the upper and lower limits of μ at l.*

THEOREM XXI. *If μ is defined on $\mathfrak{P}$ and l is a limiting element of $\mathfrak{P}$, then the following two conditions are necessary and sufficient for the existence of a finite limit of μ at l:*

1. *In every segment containing l there is a p distinct from l such that μ_p is finite.*

2. *For every e there is an $\mathfrak{S}_e$ containing l such that, if p_1 and p_2 are in $\mathfrak{S}_e$ and distinct from l, then $| \mu_{p_1} - \mu_{p_2} | \leqq e$.*

The necessity of these conditions is obvious, and the sufficiency is easily established by showing that the upper limit and the lower limit cannot be distinct.

DEFINITION 11. The function μ is *continuous* on $\mathfrak{P}$ if μ_p is finite for every p and if for every limiting element l of $\mathfrak{P}$ that lies in $\mathfrak{P}$ we have $\lim \mu_p = \mu_l$.

DEFINITION 12. The function μ is *extensibly continuous* on $\mathfrak{P}$ if μ is con-

tinuous on $\mathfrak{P}$ and if for every limiting element l of $\mathfrak{P}$ not in $\mathfrak{P}$ we have $\lim \mu_p$ existing and finite.*

THEOREM XXII. *If $\mathfrak{L}$ is the derived class of $\mathfrak{P}$ and μ is defined on $\mathfrak{P}$ and φ is defined on $\mathfrak{P} + \mathfrak{L}$ such that for every p we have $\mu_p = \varphi_p$ and for every l not in $\mathfrak{P}$ we have $\overline{\lim}\, \mu_p = \varphi_l$, then if μ is extensibly continuous on $\mathfrak{P}$, φ is extensibly continuous on $\mathfrak{P} + \mathfrak{L}$.*

Since $\mathfrak{P} + \mathfrak{L}$ is closed, on this class extensible continuity is not different from ordinary continuity. For every l and e there exists an $\mathfrak{S}_e$ containing l such that for p in $\mathfrak{S}_e$ we have $|\varphi_p - \varphi_l| \leqq e/2$. For any l_1 in $\mathfrak{S}_e$ there is an $\overline{\mathfrak{S}}_e$ containing l_1 such that for p in $\overline{\mathfrak{S}}_e$ we have $|\varphi_{l_1} - \varphi_p| \leqq e/2$, and since $\mathfrak{S}_e$ and $\overline{\mathfrak{S}}_e$ have a common p, we see from these conditions that for any q of $\mathfrak{P}$ or $\mathfrak{L}$ in $\mathfrak{S}_e$ we have $|\varphi_q - \varphi_l| \leqq e$.

6. FUNCTIONS ON A COMPOSITE RANGE.

For theorems on iterated limits we consider functions on a composite range. It is sufficient for our purpose to consider a system $(\mathfrak{Q}; B)$, composite of two systems $(\mathfrak{Q}'; B')$ and $(\mathfrak{Q}''; B'')$, each fulfilling Postulates I and II. The propositions of the present section relate to a function μ defined on a subclass $\mathfrak{P}$ of $\mathfrak{Q}$ of the form $\mathfrak{P}'\,\mathfrak{P}''$, where $\mathfrak{P}'$ and $\mathfrak{P}''$ are subclasses of $\mathfrak{Q}'$ and $\mathfrak{Q}''$ respectively. The notations l, l', l'' invariably represent limiting elements respectively of $\mathfrak{P}$, $\mathfrak{P}'$ and $\mathfrak{P}''$. The definite significance of these notations is presupposed in the statement of each theorem.

THEOREM XXIII. *If $l = l'\,l''$, then*

$$(a)\qquad \lim_{p''\to l''}\ \lim_{p'\to l'} \mu_{p'\,p''} \leqq \lim_{p''\to l''}\ \overline{\lim_{p'\to l'}} \mu_{p'\,p''} \leqq \overline{\lim_{p''\to l''}}\ \overline{\lim_{p'\to l'}} \mu_{p'\,p''},$$

$$(b)\qquad \lim_{p''\to l''}\ \lim_{p'\to l'} \mu_{p'\,p''} \leqq \overline{\lim_{p''\to l''}}\ \lim_{p'\to l'} \mu_{p'\,p''} \leqq \overline{\lim_{p''\to l''}}\ \overline{\lim_{p'\to l'}} \mu_{p'\,p''},$$

$$(c)\qquad \lim_{p\to l} \mu_p \leqq \lim_{p''\to l''}\ \lim_{p'\to l'} \mu_{p'\,p''} \leqq \overline{\lim_{p''\to l''}}\ \overline{\lim_{p'\to l'}} \mu_{p'\,p''} \leqq \overline{\lim_{p\to l}} \mu_p.$$

(*d*) *If $\lim \mu_p$ exists then*

$$\lim_{p''\to l''}\ \lim_{p'\to l'} \mu_{p'\,p''} = \lim_{p''\to l''}\ \overline{\lim_{p'\to l'}} \mu_{p'\,p''} = \lim_{p\to l} \mu_p.$$

Propositions (*a*) and (*b*) are sufficiently obvious applications of Theorem XIX. Proposition (*d*) is a corollary of (*c*), which may be established as follows: We show that the assumption that the lower limit is greater than the iterated lower limit leads to a contradiction. If we make this assumption, two cases

* In the paper on *Iterated Limits in General Analysis*, loc. cit., we show that in many applications this condition is equivalent to uniform continuity. See especially chapter IV, theorems V, XIII, XIV and XV.

may arise, the lower limit may be a finite number, or $+\infty$. Suppose

$$\lim_{p \to l} \mu_p = a, \qquad \lim_{p'' \to l''} \lim_{p' \to l'} \mu_{p'\,p''} = a - 4e.$$

Let φ be a function on $\mathfrak{P}''$ such that for every p'' we have

$$\lim_{p' \to l'} \mu_{p'\,p''} = \varphi_{p''}.$$

There is an $\mathfrak{S}_e$ containing l such that, if p is distinct from l in $\mathfrak{S}_e$, $\mu_p \geqq a - e$. This $\mathfrak{S}_e$ is of the form $\mathfrak{S}'\,\mathfrak{S}''$, and there exist p'' in $\mathfrak{S}''$ distinct from l'' such that $\varphi_{p''} \leqq a - 3e$ and p' in $\mathfrak{S}'$ distinct from l' such that $\mu_{p'\,p''} \leqq \varphi_{p''} + e$. We have, therefore, $\mu_{p'\,p''} \leqq a - 2e$, and the contradiction is found in the fact that $p = p'\,p''$ is in $\mathfrak{S}_e$. If we suppose the lower limit is $+\infty$ and the iterated lower limit is finite, similar reasoning leads to a contradiction.

THEOREM XXIV. *(a) If l' is such that for every l of the form $l = l'\,p''$ we have*

$$\lim_{p \to l} \mu_p = \varphi_{p''},$$

where φ is a finite valued function on $\mathfrak{P}''$, then φ is continuous on $\mathfrak{P}''$.

(b) If l' is such that for every l of the form $l = l'\,q''$, q'' being in $\mathfrak{P}'' + \mathfrak{Q}''$, we have

$$\lim_{p \to l} \mu_p = \varphi_{q''},$$

where φ is a finite valued function on $\mathfrak{P}'' + \mathfrak{Q}''$, then φ is extensibly continuous on $\mathfrak{P}'' + \mathfrak{Q}''$.

*(c) If μ is continuous on $\mathfrak{P}$, then, for every p', $\mu_{p'}$ is continuous on $\mathfrak{P}''$ and, for every p'', $\mu_{p''}$ is continuous on $\mathfrak{P}'$.**

(d) If μ is extensibly continuous on $\mathfrak{P}$, then, for every p', $\mu_{p'}$ is extensibly continuous on $\mathfrak{P}''$ and, for every p'', $\mu_{p''}$ is extensibly continuous on $\mathfrak{P}'$.

These propositions may be regarded as corollaries of those of the theorem just preceding.

In the remaining theorems of this section we employ notions analogous to that of uniform convergence of a sequence of functions, and to that of a family of equally continuous functions.†

DEFINITION 13. If φ is defined and finite on $\mathfrak{R}''$, a subclass of $\mathfrak{P}''$, $\mu_{p'}$ is said to *approach φ uniformly on $\mathfrak{R}''$* as p' approaches l' if for every e there exists an $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ distinct from l' and p'' is in $\mathfrak{R}''$ we have $|\mu_{p'\,p''} - \varphi_{p''}| \leqq e$. The notation for this condition is $\lim_{p' \to l'} \mu_{p'} = \varphi\,(\mathfrak{R}'')$.

* The notation $\mu_{p'}$ indicates a function on $\mathfrak{P}''$, the argument p' being fixed while p'' ranges over the class $\mathfrak{P}''$.

† Fréchet, loc. cit., p. 10.

DEFINITION 14. The function μ is *continuous on* $\mathfrak{P}'$ *uniformly on* $\mathfrak{P}''$ if for every l' in $\mathfrak{P}'$ we have $\lim_{p' \to l'} \mu_{p'} = \mu_{l'}(\mathfrak{P}'')$.

DEFINITION 15. The function μ is *extensibly continuous on* $\mathfrak{P}'$ *uniformly on* $\mathfrak{P}''$ if it is continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$ and if for every l' not in $\mathfrak{P}'$ there is a finite valued function φ on $\mathfrak{P}''$ such that we have $\lim_{p' \to l'} \mu_{p'} = \varphi(\mathfrak{P}'')$.

THEOREM XXV. *If $l = l'\, l''$ and $\mathfrak{R}''$ is the class of all elements p'' in some fixed $\mathfrak{S}''$ containing l'', then if φ is a finite valued function on $\mathfrak{P}''$ such that*

$$\lim_{p' \to l'} \mu_{p'} = \varphi(\mathfrak{R}'')$$

we have

$$(1) \qquad \lim_{p \to l} \mu_p = \lim_{p'' \to l''} \varphi_{p''}, \qquad \overline{\lim_{p \to l}}\, \mu_p = \overline{\lim_{p'' \to l''}}\, \varphi_{p''},$$

provided that in case l' is in $\mathfrak{P}'$ we have the additional hypothesis

$$\lim_{p'' \to l''} \varphi_{p''} \leqq \lim_{p'' \to l''} \mu_{l'\, p''}, \qquad \overline{\lim_{p'' \to l''}}\, \mu_{l'\, p''} \leqq \overline{\lim_{p'' \to l''}}\, \varphi_{p''},$$

and in case l'' is in $\mathfrak{P}''$ the hypothesis

$$\lim_{p'' \to l''} \varphi_{p''} \leqq \varphi_{l''} \leqq \overline{\lim_{p'' \to l''}}\, \varphi_{p''}.^*$$

Denote the lower limits in (1) by L and L''. We prove $L = L''$ for the case when l' is not in $\mathfrak{P}'$ and l'' is not in $\mathfrak{P}''$. That L is not greater than L'' follows from Theorem XXIII, (*c*). Now suppose that L'' is the greater. Consider first the case when $L'' = a$, and let $L = a - 4e$. There exists an $\mathfrak{S}''_e$ containing l'' and contained in the fixed $\mathfrak{S}''$ such that if p'' is in $\mathfrak{S}''_e$ we have $\varphi_{p''} \geqq a - e$, and there exists $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ and p'' in $\mathfrak{S}''_e$ we have $|\, \mu_{p'\, p''} - \varphi_{p''}\,| \leqq e$, whence $\mu_{p'\, p''} \geqq a - 2e$. Also, in every $\mathfrak{S}$ containing l there is a p such that $\mu_p \leqq a - 3e$. We find a contradiction when $\mathfrak{S} = \mathfrak{S}'_e\, \mathfrak{S}''_e$. Consider now $L'' = +\infty$ and $L = \underline{a}$. For a given a there is an $\mathfrak{S}''_a$ containing l'' and contained in $\mathfrak{S}''$ such that if p'' is in $\mathfrak{S}''_a$ we have $\varphi_{p''} \geqq a$, and for a given e there is an $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ and p'' is in $\mathfrak{S}''_a$ we have $|\, \mu_{p'\, p''} - \varphi_{p''}\,| \leqq e$, whence $\mu_{p'\, p''} \geqq a - e$. Also, in every $\mathfrak{S}$ containing l there is a p such that $\mu_p \leqq \underline{a} + e$. We find a contradiction when $\mathfrak{S} = \mathfrak{S}'_e\, \mathfrak{S}''_a$ and $a > \underline{a} + 2e$.

The equality of the upper limits for this case follows by analogy, and in case the restriction as to l is removed the additional hypotheses then provided clearly validate the conclusions.

Similar considerations lead also to the following theorem, which gives similar results from milder hypotheses.

* We omit the corresponding propositions involving uniform approach to $+\infty$ and to $-\infty$, which are stated in the American Journal of Mathematics (loc. cit., chapter III, theorem X), and are valid here.

THEOREM XXVI. *If $l = l'\,l''$ and l'' is not in $\mathfrak{P}''$, and if φ and θ are finite valued functions on $\mathfrak{P}''$, then*

(a) A sufficient condition for

$$\overline{\lim_{p \to l}}\, \mu_p \leqq \overline{\lim_{p' \to l''}}\, \varphi_{p''}$$

is that there exist an $\mathfrak{S}''$ containing l'' such that for every e there is an $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ and p'' is in $\mathfrak{S}''$ then $\mu_{p'\,p''} \leqq \varphi_{p''} + e$:

(b) A sufficient condition for

$$\underline{\lim_{p \to l}}\, \mu_p \geqq \underline{\lim_{p'' \to l''}}\, \theta_{p''}$$

is that there exist an $\mathfrak{S}''$ containing l'' such that for every e there is an $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ and p'' is in $\mathfrak{S}''$ then $\mu_{p'\,p''} \geqq \theta_{p''} - e$.

THEOREM XXVII. *If $l = l'\,l''$ and $\mathfrak{S}''$ contains l'' and $\mathfrak{R}''$ consists of all elements p'' in $\mathfrak{S}''$, and if in every $\mathfrak{S}'$ containing l' there is a p' distinct from l' such that, for some a,*

$$\lim_{p'' \to l''} \mu_{p'\,p''} = a,$$

and if φ is a finite valued function on $\mathfrak{P}''$ such that

$$\lim_{p' \to l'} \mu_{p'} = \varphi\,(\mathfrak{R}''),$$

then there is an a_1 such that

$$\lim_{p''_i \to l''} \varphi_{p''} = a_1.$$

For by the uniform approach to φ we have for every e an $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ distinct from l' and if p'' is in $\mathfrak{S}''$ then

$$|\,\mu_{p'\,p''} - \varphi_{p''}\,| \leqq e/3. \tag{1}$$

Also there is a p'_1 in $\mathfrak{S}'_e$ such that, taking account of Theorem XXI, there exists an $\mathfrak{S}''_1$ containing l'' such that if p''_1 and p''_2 are in $\mathfrak{S}''_1$ and distinct from l'' we have

$$|\,\mu_{p_1'\,p_1''} - \mu_{p_1'\,p_2''}\,| \leqq e/3. \tag{2}$$

Take $\mathfrak{S}''_e$ common to $\mathfrak{S}''$ and $\mathfrak{S}''_1$, then if p''_1 and p''_2 are in $\mathfrak{S}''_e$ we have from (1) and (2) the condition $|\,\varphi_{p_1''} - \varphi_{p_2''}\,| \leqq e$, and since condition (1) of Theorem XXI is obviously fulfilled by φ at l'', we have the desired conclusion.

The two following theorems are easily deduced from the preceding. The propositions are true either with or without the words in brackets.

THEOREM XXVIII. *If l' is such that in every $\mathfrak{S}'$ containing l' there is a p' distinct from l' such that $\mu_{p'}$ is [extensibly] continuous on $\mathfrak{P}''$, and if φ is a*

finite valued function on $\mathfrak{P}''$ *such that*

$$\lim_{p' \to l'} \mu_{p'} = \varphi(\mathfrak{P}''),$$

then φ *is* [*extensibly*] *continuous on* $\mathfrak{P}''$.

THEOREM XXIX. *If* μ *is* [*extensibly*] *continuous on* $\mathfrak{P}'$ *uniformly on* $\mathfrak{P}''$, *and if for every* p' $\mu_{p'}$ *is* [*extensibly*] *continuous on* $\mathfrak{P}''$, *then* μ *is* [*extensibly*] *continuous on* $\mathfrak{P}$.

7. THEOREMS REQUIRING A THIRD POSTULATE.

Consider a class $\mathfrak{Q}$ consisting of all integers, positive and negative. Let $B_{q_1 q_2 q_3}$ hold if q_1 and q_3 are integral multiples of 4 and q_2 is between them in the usual sense. The system $(\mathfrak{Q}; B)$ fulfils Postulates I and II, but clearly violates several conditions usually considered as fundamental in the theory of limits. For example, the proposition that no finite class has a limiting element does not hold, since any class containing two successive integers has one of these as a limiting element. Again, the proposition that a sequence can have but one limit does not hold, for the sequence 2, 2, 2 . . . has the limits 1, 2, and 3. These conditions and certain others that may be desirable in extending a theory of limits are fulfilled in a system $(\mathfrak{Q}; B)$ restricted by the following postulate.

POSTULATE III. *If* q_1 *and* q_2 *are distinct elements, each contained in some segment, then there exist segments,* $\mathfrak{S}_1$ *containing* q_1 *and* $\mathfrak{S}_2$ *containing* q_2, *such that* $\mathfrak{S}_1$ *and* $\mathfrak{S}_2$ *contain no common element.*

This postulate, like the first two, is invariant under composition of systems if we exclude the trivial case in which the composite class contains no segment. Also it is obviously fulfilled by any simply ordered class, if the relation B plays the rôle of betweenness.

On the basis of the three postulates our definition of limiting element is found to fulfil the four conditions regarded as fundamental by F. Riesz.* The first is expressed in Theorem III, and the remaining conditions are embodied in the propositions of the following theorem.

THEOREM XXX. (*a*) *If* $\mathfrak{P} = \mathfrak{P}_1 + \mathfrak{P}_2$ *then every limiting element of* $\mathfrak{P}$ *is a limiting element of* $\mathfrak{P}_1$ *or of* $\mathfrak{P}_2$.

(*b*) *No finite class has a limiting element.*

(*c*) *Each limiting element of a class* $\mathfrak{P}$ *is uniquely determined by the totality of all subclasses of* $\mathfrak{P}$ *of which it is a limiting element.*

Proposition (*a*) is seen to depend only on Postulate II. As to (*b*), if $\mathfrak{P}$ consists of a finite set, $p_1, p_2, \cdots, p_n$, then for any q there exist, by Postulates I and III, segments $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n$ containing q such that if $p_i \neq q$ the seg-

* Loc. cit.

ment $\mathfrak{S}_i$ does not contain p_i. By Postulate II we secure a segment $\mathfrak{S}$ contained in all the $\mathfrak{S}_i$ but containing q. Since $\mathfrak{S}$ does not contain any p distinct from q, q is not a limiting element of $\mathfrak{P}$. To prove (c), let q_1 and q_2 be two distinct elements, then by Postulates I and III there exist $\mathfrak{S}_1$ containing q_1 and $\mathfrak{S}_2$ containing q_2 such that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ have no common elements. If q_1 and q_2 are limiting elements of $\mathfrak{P}$, they are limiting elements respectively of the two subclasses $\mathfrak{P} - \mathfrak{S}_2$ and $\mathfrak{P} - \mathfrak{S}_1$, but q_1 is not a limiting element of $\mathfrak{P} - \mathfrak{S}_1$, nor q_2 of $\mathfrak{P} - \mathfrak{S}_2$.

THEOREM XXXI. *A sequence cannot have two distinct limits.*

Theorems VI, VII and XXXI show that our definition of limit of a sequence fulfils the conditions used by M. Fréchet on page 5 of his thesis. We can now prove the two following theorems, the first stated after the manner of a theorem by Fréchet, the second after a theorem by Riesz.

THEOREM XXXII. *If every member of a sequence of subclasses of a compact class is closed, contains the succeeding member and contains at least one element, then there is an element common to all members of the sequence.*

THEOREM XXXIII. *If every member of a sequence of infinite subclasses of a compact class contains the succeeding member, the members of the sequence have a common limiting element.*

To prove this last theorem let $\{ \mathfrak{P}_n \}$ be such a sequence of subclasses and let $\{ q_n \}$ be a sequence of distinct elements such that, for every n, q_n is in $\mathfrak{P}_n$. The class $\mathfrak{P}$ of all elements in the sequence $\{ q_n \}$ is infinite and has at least one limiting element q. This q is a limiting element of every class of the sequence $\{ \mathfrak{P}_n \}$, by Theorem XXX; for, any given member of the sequence of classes contains all but a finite number of elements of the sequence of elements.

Theorem XXXII, which does not follow from the work of Fréchet because of the difference in definitions of limiting elements, is an immediate corollary of the theorem just proved.

The three postulates are sufficient also to secure the following form of the Heine-Borel theorem.

THEOREM XXXIV. *If $\{ \mathfrak{P}_n \}$ is a sequence of subclasses of Ω such that every element of a certain class $\mathfrak{P}$ which is compact and closed is interior to at least one of the $\mathfrak{P}_i$, then there is a finite set of the $\mathfrak{P}_i$ such that every element of $\mathfrak{P}$ is interior to at least one class of the finite set.*

Suppose the theorem is untrue. Then there exists a sequence $\{ p_m \}$ of distinct elements of $\mathfrak{P}$ such that p_m is not interior to $\mathfrak{P}_n$ if $n < m$. The class $\overline{\mathfrak{P}}$ of all elements of the sequence $\{ p_m \}$ has a limiting element p, which is in $\mathfrak{P}$ and therefore interior to some $\mathfrak{P}_i$, say $\mathfrak{P}_{n_1}$. Let $\overline{\mathfrak{P}} = \mathfrak{P}_1 + \mathfrak{P}_2$, where $\mathfrak{P}_1$ is the part interior to $\mathfrak{P}_{n_1}$, and $\mathfrak{P}_2$ is the remainder. $\mathfrak{P}_1$ is finite and has no limiting element. There is a segment $\mathfrak{S}$ containing p and contained in $\mathfrak{P}_{n_1}$,

therefore containing no element of $\mathfrak{P}_2$, so that p is not a limiting element of $\mathfrak{P}_2$, and consequently not of $\overline{\mathfrak{P}}$. Our supposition is therefore false.

The proof of the following theorem, which, in view of Theorem XXX (a), implies Theorem I, is obvious.

THEOREM XXXV. *The derived class of any class is closed.*

To the list of theorems rendered valid by the addition of the third postulate may be added the following propositions on functions.

THEOREM XXXVI. (a) *If $\mathfrak{P}$ is compact and μ is extensibly continuous on $\mathfrak{P}$, then μ is finitely bounded on $\mathfrak{P}$.*

(b) *If $\mathfrak{P}$ is compact, then there exist q_1 and q_2 such that the the least upper bound of μ on $\mathfrak{P}$ is either μ_{q_1} or $\overline{\lim_{p \to q_1}} \mu_p$, and the greatest lower bound of μ on $\mathfrak{P}$ is either μ_{q_2} or $\underline{\lim_{p \to q_2}} \mu_p$.*

(c) *If $\mathfrak{P}$ is compact and closed and μ is continuous on $\mathfrak{P}$, then there exist p_1 and p_2 such that the least upper bound of μ on $\mathfrak{P}$ is μ_{p_1} and the greatest lower bound of μ on $\mathfrak{P}$ is μ_{p_2}.*

Proof of (a). Suppose that μ is not bounded from $+\infty$, then it can easily be shown that there exists a sequence $\{ p_n \}$ of distinct elements such that

$$\lim_{n \to \infty} \mu_{p_n} = +\infty.$$

The elements of this sequence constitute a class having a limiting element, say l. Since μ is extensibly continuous, there is a segment $\mathfrak{S}$ containing l such that for some a and for every p in $\mathfrak{S}$ distinct from l we have $\mu_p < a$. Since $\mathfrak{S}$ must contain an infinite number of elements of the sequence $\{ p_n \}$, we reach a contradiction. Similarly it may be shown that μ is bounded from $-\infty$.

Proof of (b). We consider the cases when the least upper bound of μ on $\mathfrak{P}$ is finite and $+\infty$. In either case there exists an element p such that μ_p is this least upper bound, or there is a sequence $\{ p_n \}$ of distinct elements such that the limit of μ_{p_n} is this least upper bound. If the latter situation prevails, then this sequence gives rise to a limiting element l, and the assumption that the upper limit of μ at l is different from the least upper bound leads obviously to a contradiction. Thus if not a p, then an l will serve as the q_1 required. Similarly we may show the existence of a q_2 meeting the requirements of the theorem.

Proposition (c) is a corollary of (b).

THEOREM XXXVII. *If μ is finitely bounded on $\mathfrak{P}$, then, for a fixed positive a, the set of elements l at which*

$$\overline{\lim_{p \to l}} \mu_p - \underline{\lim_{p \to l}} \mu_p \geqq a$$

is closed.

Suppose every segment containing l contains an l_1 for which the upper and lower limits of μ differ by a quantity greater than or equal to a. Let the upper limit and lower limit at l be respectively $\bar{a}$ and $\underline{a}$, and suppose that $\bar{a} - \underline{a} < a$. Take e as one sixth of $a - \bar{a} + \underline{a}$, then there is an $\mathfrak{S}_e$ containing l such that if p is in $\mathfrak{S}_e$ and distinct from l we have $\underline{a} - e \leqq \mu_p \leqq \bar{a} + e$, and therefore if p_1 and p_2 are in $\mathfrak{S}_e$ and distinct from l we have $|\mu_{p_1} - \mu_{p_2}| \leqq a - 4e$. But if l_1 is in $\mathfrak{S}_e$ and such that the upper limit and lower limits of μ at l_1 are respectively $\bar{a}_1$ and $\underline{a}_1$, where $\bar{a}_1 - \underline{a}_1 \geqq a$, then we may take p_1 and p_2 in $\mathfrak{S}_e$ distinct from l such that $\mu_{p_1} \geqq \bar{a}_1 - e$ and $\mu_{p_2} \leqq \underline{a}_1 + e$, whence $|\mu_{p_1} - \mu_{p_2}| \geqq a - 2e$, and this affords a contradiction.

8. A SEQUENTIAL THEORY FOR A SYSTEM $(\mathfrak{Q}; B)$.

In the theory of point sets a limiting point of a set may be defined as a point every neighborhood of which contains a point of the set distinct from itself, or as a point which is the limit of a sequence of distinct points of the set. The notion "limit of a sequence" being defined in terms of "neighborhood," the two definitions are known to be equivalent. In the domain of general analysis, where elements of arbitrary character are considered, it may happen that, even though the notions involved in both definitions of limiting element are well defined, and even though either definition may lead to a theory of very considerable extent, the two theories need not be equivalent.

The definition of limiting element we have employed in the foregoing sections is an instance of a "neighborhood" definition, and the theory developed is in terms of "segment," which is an instance of the general notion of neighborhood. The theory might be characterized as a "neighborhood theory" for a system $(\mathfrak{Q}; B)$. But since the notion "limit of a sequence" is defined (Def. 7), we might adopt a "sequential" definition of limiting element and develop a "sequential theory" for a system of this type. Fortunately, however, this theory becomes a special case of a general theory receiving consideration in papers by M. Fréchet,* E. R. Hedrick,† and T. H. Hildebrandt.‡ Our definition of limit of a sequence satisfies all conditions used by Fréchet and Hildebrandt in the purely sequential parts of their papers. Hildebrandt shows that for many of the theorems none of these conditions are necessary. It is obviously true, also, that in this sequential theory we should have the theorem "every derived class is closed," which in the neighborhood theory must be stated, "if every element of a sequence whose limit is q is a limit of a

* The first part of the paper by M. Fréchet referred to above is purely "sequential" in character.

† *On Properties of a Domain for which any Derived Set is Closed*, these Transactions, vol. 12 (1911), pp. 285–294.

‡ *A Contribution to the Foundations of Frechet's Calcul Fonctionnel*, American Journal of Mathematics, vol. 34 (1912), p. 237.

sequence of distinct elements of $\mathfrak{P}$, q is a limit of a sequence of distinct elements of $\mathfrak{P}$." This proposition, with the conditions on "limit" postulated by Fréchet, forms the basis for the first part of the paper by Hedrick.

It appears, then, that a considerable body of "sequential theory" is available for a system $(\mathfrak{Q}; B)$ subject to our three postulates. We show by the following two examples that for such a system theorems of like form in the sequential and neighborhood theories need not be equivalent theorems.

EXAMPLE 4. Let $\mathfrak{Q}$ be the class of all infinite sequences of real numbers. For any three such sequences, $q_1 = q_1^1, q_1^2, q_1^3, \cdots$, $q_2 = q_2^1, q_2^2, q_2^3, \cdots$; and $q_3 = q_3^1, q_3^2, q_3^3, \cdots$, let the relation $B_{q_1 q_2 q_3}$ hold if and only if, for every value of i, we have either $q_1^i < q_2^i < q_3^i$ or $q_3^i < q_2^i < q_1^i$. The three postulates are obviously fulfilled by this system $(\mathfrak{Q}; B)$. Let $\mathfrak{P}$ be the subclass consisting of all infinite sequences of positive real numbers. The element $q = 0, 0, 0, \cdots$ is a limiting element of $\mathfrak{P}$, but q is not the limit of any sequence of elements of $\mathfrak{P}$. For suppose the sequence $\{ p_n \}$ to have the limit q, then the element $p = p^1, p^2, p^3, \cdots$, where $p^i = p_i^i$, with any sequence of negative real numbers, clearly determines a segment such that no element of the sequence $\{ p_n \}$ is in the segment.

The system $(\mathfrak{Q}; B)$ of this example may be regarded as a composite system, composite of a denumerable infinitude of systems $(\mathfrak{Q}^i; B^i)$, in each of which $\mathfrak{Q}^i$ is the class of all real numbers and B^i is the ordinary relation of betweenness.

EXAMPLE 5. Let $\mathfrak{K}$ be a non-denumerable class of elements k to which a definite normal order has been assigned, i. e., let $\mathfrak{K}$ be well-ordered but not capable of a one-to-one correspondence with the class of natural numbers or a subclass of it.* Let $\mathfrak{P}$ consist of all denumerable subclasses of $\mathfrak{K}$, and let $\mathfrak{Q}$ consist of all elements of $\mathfrak{P}$ together with all real numbers except those between zero and one. For this class $\mathfrak{Q}$ we define first a relation of precedence $(<)$. With respect to the real numbers in $\mathfrak{Q}$ the relation precedes shall have the usual significance. Every element p of $\mathfrak{P}$ shall precede all positive real numbers in $\mathfrak{Q}$, and shall follow all other real numbers in $\mathfrak{Q}$. Of two distinct elements, p_1 and p_2, of $\mathfrak{P}$, that one precedes which contains the first element k, according to the normal order of $\mathfrak{K}$, which is in one and only one of the two subclasses p_1 and p_2 of $\mathfrak{K}$. The relation of precedence so defined is transitive, so that the conditions for linear order are fulfilled. It may also be shown that $\mathfrak{Q}$ is dense, and that the pseudo-archimedean postulate used by Veblen is fulfilled.†

Let $B_{q_1 q_2 q_3}$ hold if and only if $q_1 < q_2 < q_3$ or $q_3 < q_2 < q_1$, then it is clear that all three of our postulates are fulfilled by the system $(\mathfrak{Q}; B)$.

* Compare O. Veblen, *Definition in Terms of Order Alone in the Linear Continuum and in Well-ordered Sets*, these Transactions, vol. 6 (1905), p. 169.

† O. Veblen, loc. cit., p. 165.

Now the element $q_0=0$ is a limiting element of the subclass $\mathfrak{P}$. For, given any p, a denumerable subclass of $\mathfrak{K}$, we may take p_1 as a subclass of $\mathfrak{K}$ including every k in p and one additional k; then $p_1 < p$. But no sequence $\{ p_n \}$ of elements of $\mathfrak{P}$ can have the limit q_0; for the least common superclass of the subclasses p_i that occur in the sequence is denumerable, and an element p consisting of this and one k in addition will precede every p_i.

It thus appears that the equivalence of the "neighborhood" and the "sequential" definitions of limiting element involves serious restrictions on a system $(\mathfrak{Q}; B)$, in addition to our postulates, and even in addition to the assumption that $\mathfrak{Q}$ is simply ordered, dense, and satisfies the pseudo-archimedean postulate.

9. The sequential property.

We shall say that a subclass $\mathfrak{P}$ of $\mathfrak{Q}$ has the *sequential property* if every subclass of $\mathfrak{P}$ that has a limiting element q contains a sequence of distinct elements whose limit is q. This property is definite for any subclass of a class $\mathfrak{Q}$ for which the two notions, "limiting element of a subclass" and "limit of a sequence" are defined.* In particular, if the idea of neighborhood of an element is available for a class $\mathfrak{Q}$, then definitions of the form of definitions 2 and 7 of section 2 afford a definite sequential property.

Relative to a system of the type $(\mathfrak{Q}; B)$ we consider the following postulates.

Postulate IV. *For every q there exists a sequence $\{ \mathfrak{S}_n \}$ of segments containing q such that, for every segment $\mathfrak{S}$ containing q, there is an $n_{\mathfrak{S}}$ such that, if $n > n_{\mathfrak{S}}$, $\mathfrak{S}_n$ is a subclass of $\mathfrak{S}$.*

Postulate IV′. *For every q there exists a sequence $\{ \mathfrak{S}_n \}$ of segments containing q and having no other common element.*

Each of these postulates is invariant under composition, provided only that the composite system shall possess at least one element. If $\mathfrak{Q}$ is simply ordered and B is the ordinary relation of betweenness, then these two postulates are equivalent, and each is equivalent to the assumption that $\mathfrak{Q}$ has the sequential property. Without the assumption of linear order, it is easily seen that Postulate IV implies I and II, and that III and IV together imply IV′. The illustration at the beginning of section 7 is an instance of a system fulfilling I, II and IV, but not III or IV′. Example 4 fulfils I, II, III and IV′ but not IV. It may be shown that in any system fulfilling Postulates III and IV the class $\mathfrak{Q}$ has the sequential property, and therefore that for such a system the sequential theory and the neighborhood theory are equivalent.

Systems which satisfy Postulates III and IV, and therefore I, II and IV′, are of very frequent occurrence. In the definition of the linear continuum

* The question of a sequential property is obviously trivial in case the sequential definition of limiting element is employed. It is conceivable that various definitions of limiting element may be given, without the use of sequence or neighborhood.

given by O. Veblen, condition A (linear order) and condition E (postulate of uniformity), the latter so modified as to be effective without B, C, and D, and weakened by the omission of properties 1 and 3, are sufficient to secure such a system, where the relation B is ordinary betweenness.* This modified E may be replaced by the stronger "postulate of linearity" which is due to G. Cantor and occurs in E. V. Huntington's exposition of the modern theory of the continuum.† This postulate would require that there exist a denumerable subclass $\mathfrak{R}$ of $\mathfrak{Q}$ such that every segment must contain at least one element of $\mathfrak{R}$, and that $\mathfrak{Q}$ shall be dense.

Turning now to a consideration of the sequential property in relation to a general notion of "neighborhood," of which the notion segment is obviously a special case, we observe that this property is present in systems of much more general character than a system $(\mathfrak{Q}; B)$ fulfilling Postulates III and IV. In the paper on iterated limits in general analysis‡ we take the notion "neighborhood" as undefined, that is, a relation T between subclass and element is undefined but subject to postulates that validate a theory in which a class having the relation T to an element serves as a neighborhood of that element. The postulates, stated in the second chapter of the paper, pertain to a system $(\mathfrak{P}; \mathfrak{U}; T)$, in which $\mathfrak{P}$ is a class of elements, $\mathfrak{U}$ is a class of ideal elements, and T is a relation between subclasses of $\mathfrak{P}$ and elements of the class $\mathfrak{Q} = \mathfrak{P} + \mathfrak{U}$. We obtain a special instance of a system of this type from a system $(\mathfrak{Q}; B)$ if we set $\mathfrak{P} = \mathfrak{Q}$, let $\mathfrak{U}$ be the null class, and let a class $\mathfrak{R}$ have the relation T to an element p if and only if $\mathfrak{R}$ is a segment containing p. Postulates III and IV on the system $(\mathfrak{Q}; B)$ are sufficient to secure the five postulates used for the system $(\mathfrak{P}; \mathfrak{U}; T)$, in fact, all but the third and fifth result from the type of $(\mathfrak{Q}; B)$, without the use of postulates. Now Theorem IV of the second chapter of the paper cited shows that every subclass has the sequential property; and the proof of this theorem involves only Postulates (I), (II), (III) and (V), while (II) may be dispensed with in case no ideal elements are present.

Other systems in terms of which a system $(\mathfrak{P}; \mathfrak{U}; T)$ may be set up in such manner as to satisfy Postulates (I)–(V), and which, therefore, secure the sequential property, are discussed in the fourth chapter of the same paper. These include cases of neighborhood defined in terms of *voisinage* as used by M. Fréchet,§ and in terms of suitably conditioned K relations of the type used by E. H. Moore‖ and T. H. Hildebrandt.¶

* O. Veblen, loc. cit., p. 166.

† E. V. Huntington, Annals of Mathematics, ser. 2, vol. 7 (1905), p. 15.

‡ Loc. cit.

§ Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), p. 17.

‖ *Introduction to a Form of General Analysis*, part II, p. 125 et seq.

¶ Loc. cit., p. 242.

In his paper on properties of a domain for which any derived set is closed, E. R. Hedrick* reverses the process we have used and defines "neighborhood" in terms of "limit." Assuming a limit relation fulfilling the conditions prescribed by Fréchet, and using a sequential definition of limiting element, he postulates that the fundamental domain is compact and that the derived class of every subclass is closed. He later assumes the "enclosable property," which affords a set of subclasses subject to conditions that validate their use in the sense of neighborhood. The type of neighborhood thus available may be used to set up a new definition of limiting element, in which case the fundamental domain is found to have the sequential property, so that the sequential theory he secures for the domain thus conditioned is equivalent to a neighborhood theory based, in the usual manner, on the type of neighborhood his assumptions afford. In fact, a domain with the enclosable property leads, in a very obvious manner, to a system $(\mathfrak{P}; \mathfrak{U}; T)$ fulfilling all five of the postulates we have used for such a system.

UNIVERSITY OF MISSOURI,
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* Loc. cit., p. 285.

THE SYMBOLICAL THEORY OF FINITE EXPANSIONS*

BY

O. E. GLENN

INTRODUCTION.

The first two sections of this paper are the result of an attempt to identify, and state in general terms, certain fundamental common properties of a large number of known types of finite expansions of polynomials in p variables. The number of such expansions which are known and available for such a synthesis is of course large, and only the most salient properties can be comprised under one general theory. The principles proved in these preliminary sections are subjected to verification when it is shown that they hold for a number of new expansions. These are derived in sections 3 to 6 inclusive. They are of considerable interest in themselves, apart from the general theory, to the fundamental laws of which they conform.

The Aronhold symbolical notation† for algebraic forms subject to linear transformations constitutes the basis of the general methods of the paper.

To state briefly our initial point of view; if a function f can be given an expression of the form

$$f = B_0 A_1^m + mB_1 A_1^{m-1} A_2 + \binom{m}{2} B_2 A_1^{m-2} A_2^2 + \cdots + B_m A_2^m,$$

it can be represented symbolically as an mth power, say

$$f = \Xi_A^m = (\Xi_1 A_1 + \Xi_2 A_2)^m = \Xi_1^m A_1^m + m \Xi_1^{m-1} \Xi_2 A_1^{m-1} A_2 + \cdots.$$

But with $\nabla_1 = \Xi_2 (\partial / \partial \Xi_1)$ this may be written

$$f = \Xi_1^m A_1^m + \frac{\nabla_1 \Xi_1^m}{\lfloor 1} A_1^{m-1} A_2 + \frac{\nabla_1^2 \Xi_1^m}{\lfloor 2} A_1^{m-2} A_2^2 + \cdots + \frac{\nabla_1^m \Xi_1^m}{\lfloor m} A_2^m.$$

Now the symbol Ξ_1^m equals B_0. If it prove possible to find a non-symbolical operator ∇ equivalent (§ 1) to ∇_1, then we shall have expressed f in the funda-

* Presented to the Society, April, 1911, and March, 1913.

† Aronhold, Journal für Mathematik, vol. 55 (1858), p. 97.

mental, non-symbolical form

$$f = B_0 A_1^n + \frac{\nabla B_0}{\lfloor 1} A_1^{n-1} A_2 + \frac{\nabla^2 B_0}{\lfloor 2} A_1^{n-2} A_2^2 + \cdots + \frac{\nabla^m B_0}{\lfloor m} A_2^n .$$

This is what is actually done in what follows.

§ 1. Fundamental Lemma.

Definitions. The general term *function* will be used to indicate an algebraic form in p variables.

By an *expansion* of a function we shall mean a representation of it as a finite series proceeding by a law in powers of one or more arguments, with coefficients dependent upon the coefficients, and possibly upon the variables also (§3), of the original function.

Equivalent operators. In connection with the familiar symbolical representation of a binary form

$$f = a_x^n = (\alpha_1 x_1 + \alpha_2 x_2)^n = \alpha_1^n x_1^n + n\alpha_1^{n-1} \alpha_2 x_1^{n-1} x_2 + \cdots$$
$$= a_0 x_1^n + n a_1 x_1^{n-1} x_2 + \cdots ,$$

we note that the expressions $\alpha_1^n\,(=a_0)$, $\alpha_1^{n-1}\alpha_2\,(=a_1)$, $\cdots$ and linear combinations of these, are the only expressions in the symbolical α_i, which are *defined* in terms of the actual coefficients. We call an expression, as $I = \alpha_1^n + p\alpha_1^{n-2}\alpha_2^2$, the *symbolical equivalent* of the corresponding expression in the a's, $J = a_0 + pa_2$.

The differential operator $\alpha_2\,(\partial/\partial\alpha_1)$ is such that when

$$\text{(i)} \qquad \alpha_2 \frac{\partial}{\partial \alpha_1} I = K ,$$

and I is a defined expression, then K is a defined expression. Let the non-symbolical equivalent of K be L, that of I being J as illustrated above. Then the non-symbolical operator

$$O = n a_1 \frac{\partial}{\partial a_0} + (n-1)\, a_2 \frac{\partial}{\partial a_1} + \cdots + a_n \frac{\partial}{\partial a_{n-1}}$$

has the property that*

* For example when I is the particu'ar expression given above, equations (i), (ii) are respectively

$$\alpha_2 \frac{\partial}{\partial \alpha_1} (\alpha_1^n + p\alpha_1^{n-2}\alpha_1^n) = n\alpha_1^{n-1}\alpha_2 + p\,(n-2)\,\alpha_1^{n-3}\alpha_2^3 ,$$

$$O\,(a_0 + pa_2) = na_1 + p\,(n-2)\,a_3 .$$

The operands on the left are equivalent, and the right-hand members are also equivalent expressions.

(ii) $$O J = L.$$

Two operators, one symbolical and the other non-symbolical, related like $\alpha_2 (\partial / \partial \alpha_1)$ and O will be called equivalent or isomorphic operators. We shall refer to this important property of equivalence of two operators as the property (A), and we now state it in such general terms as will completely define it for all the symbolical representations of functions that are employed in the sequel.

(A) Let there be given a definite symbolical representation of a function f. A symbolical derivative operator on an expression in the defined combinations of the symbols, which produces a result containing the symbols in defined combinations only, has corresponding to it a non-symbolical derivative operator which carries the non-symbolic equivalent of the first expression into that of the second.

Finite Expansions. Any finite expansion of a function f can be made the basis of a symbolical representation of that function. For, if there exists an expansion of f in powers of a given argument A,

$$f = B_0 + \binom{m}{1} B_1 A + \binom{m}{2} B_2 A^2 + \cdots + B_m A^m,$$

then f may be represented as the mth power of a purely symbolical binomial expression

(1) $$\Xi_1 + \Xi_2 A.$$

That is, if we write $\Xi_{(A)}$ as an abbreviation of this non-homogeneous (as to A) expression, then

(1′) $$f = \Xi_{(A)}^m = (\Xi_1 + \Xi_2 A)^m = \Xi_1^m + m\Xi_1^{m-1}\Xi_2 A + \cdots = B_0 + mB_1 A + \cdots.$$

Then the only defined expressions in the symbols are

$$\Xi_1^{m-k}\Xi_2^k \ (= B_k) \qquad (k = 0, 1, \cdots, m),$$

and linear combinations of these, where, according to our definition of an expansion, the B_k are non-symbolical, determinate expressions in the coefficients and variables of f.

Symbolical basis. In the case of every finite expansion with which we shall deal, (i) the property (A) holds for both operators

$$\nabla_1 = \Xi_2 \frac{\partial}{\partial \Xi_1}, \qquad \nabla_2 = \Xi_1 \frac{\partial}{\partial \Xi_2},$$

and (ii) the non-symbolical equivalents of ∇_i can actually be found, expressed, not in terms of the unknown B_k, but in terms of the coefficients of the known function f, and the known argument A.

When (i), (ii) have actually been verified we say that the existence of a symbolical basis of the expansion, viz.,

$$\Xi_{(A)} = \Xi_1 + \Xi_2 A,$$

has been verified.

Fundamental lemma. For an expansion for which a symbolical basis exists we now prove a lemma. Evidently

$$\nabla_1 \Xi_1^{m-k} \Xi_2^k = (m-k)\, \Xi_1^{m-k-1} \Xi_2^{k+1} \qquad (k = 0, 1, \cdots, m).$$

Hence (1′) gives

$$f = \Xi_1^m + \nabla_1 \Xi_1^m A + \frac{\nabla_1^2 \Xi_1^m}{\lfloor 2} A^2 + \cdots. \tag{2′}$$

Write ∇ for the non-symbolical equivalent of ∇_1, according to the property (A), and set $\Phi = B_0 (= \Xi_1^m)$. Then from (2′) we have

Lemma. *There exists a function Φ and a differential operator ∇, capable of being represented symbolically by the linear operator*

$$\nabla_1 = \Xi_2 \frac{\partial}{\partial \Xi_1},$$

such that the explicit expansion of f is

$$f = \Phi + \frac{\nabla \Phi}{\lfloor 1} A + \frac{\nabla^2 \Phi}{\lfloor 2} A^2 + \cdots + \frac{\nabla^m \Phi}{\lfloor m} A^m. \tag{2}$$

If a symbolical basis be assumed in homogeneous form, or, what amounts to the same thing, if f be assumed to have a homogeneous expansion in two arguments A_1, A_2 for which a symbolical basis exists in the form

$$\Xi_A = \Xi_1 A_1 + \Xi_2 A_2,$$

then

$$f = \Xi_A^m = \Xi_1^m A_1^m + m\, \Xi_1^{m-1} \Xi_2 A_1^{m-1} A_2 + \cdots = \Phi_1 A_1^m + \cdots + \Phi_2 A_2^m. \tag{3′}$$

This is the same as (2) when $A_1 = 1$. But, in general, we have

$$\frac{\nabla_1^r \Xi_1^m}{\lfloor r} = \frac{\nabla_2^{m-r} \Xi_2^m}{\lfloor m-r} = \frac{1}{2}\left(\frac{\nabla_1^r \Xi_1^m}{\lfloor r} + \frac{\nabla_2^{m-r} \Xi_2^m}{\lfloor m-r} \right) = \binom{m}{r} \Xi_1^{m-r} \Xi_2^r.$$

Thus the third expression can be used in place of the general coefficient in (3′). Let ∇_i' be the non-symbolical equivalent of ∇_i $(i = 1, 2)$. Then we get from (3′) an expansion formula, more general, and more symmetrical than (2), viz.,

$$f = \Phi_1 A_1^m + \frac{1}{2}\left(\frac{\nabla_1' \Phi_1}{\lfloor 1} + \frac{\nabla_2'^{m-1} \Phi_2}{\lfloor m-1} \right) A_1^{m-1} A_2 + \cdots + \frac{1}{2}\left(\frac{\nabla_1'^{m-1} \Phi_1}{\lfloor m-1} + \frac{\nabla_2' \Phi_2}{\lfloor 1} \right) A_1 A_2^{m-1} + \Phi_2 A_2^m. \tag{3}$$

Such an expansion will be completely given by two non-symbolical operators ∇_i' and two pre-determined functions Φ_i $(i = 1, 2)$.

The importance of a general formula like (3) evidently depends upon the number of expansions which can be found to conform to its laws. We proceed to point out the fact that many of the known finite expansions are special cases of (2) or (3).

(*a*) Any binary form $f(x_1, x_2) = a_0 x_1^m + ma_1 x_1^{m-1} x_2 + \cdots$, when transformed by

$$x_1 = \lambda_1 x_1' + \mu_1 x_2', \qquad x_2 = \lambda_2 x_1' + \mu_2 x_2',$$

gives a result $f_1(x_1', x_2')$ which is an expansion of f in the arguments x_i'. It is then a case of (3) where now $A_i = x_i'$, and

$$\Phi_1 = f(\lambda_1, \lambda_2), \qquad \Phi_2 = f(\mu_1, \mu_2),$$

$$\nabla_1' = \mu_1 \frac{\partial}{\partial \lambda_1} + \mu_2 \frac{\partial}{\partial \lambda_2}, \qquad \nabla_2' = \lambda_1 \frac{\partial}{\partial \mu_1} + \lambda_2 \frac{\partial}{\partial \mu_2}.$$

(*b*) Any covariant of $f(x_1, x_2)$, say $C = C_0 x_1^w + \cdots$, considered as an expansion in x_1, x_2 is a case of (3). Here $\Phi_1 = C_0$, the seminvariant leading coefficient; $\Phi_2 = C_w$. Or $x_1^{-w} C$ is a case of (2), where $A = x_1 / x_2$. In either case

$$\nabla_1' = ma_1 \frac{\partial}{\partial a_0} + (m-1)\, a_2 \frac{\partial}{\partial a_1} + \cdots + a_m \frac{\partial}{\partial a_{m-1}},$$

$$\nabla_2' = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + \cdots + ma_{m-1} \frac{\partial}{\partial a_m}.$$

(*c*) If $\varphi = (a_0, a_1, \cdots)(x_1, x_2)^m$ goes into $(a_0', a_1', \cdots)(x_1', x_2')^m$ under the special transformation $x_1 = x_1' + Ax_2'$, $x_2 = x_2'$, and $I(a_0, a_1, \cdots)$ is any rational integral function* of $a_0, a_1, \cdots$, then the expansion of $I(a_0', a_1', \cdots)$ in powers of A is a case of (2), where $f = I(a_0', \cdots)$, $\Phi = I(a_0, a_1, \cdots)$, and ∇ is the preceding ∇_2'.

(*d*) Taylor's expansion of a non-homogeneous polynomial $f(x)$ in powers of $x - a$ is a case of (2), where

$$\Phi = f(a), \qquad \nabla = \frac{\partial}{\partial a}.$$

(*e*) As a proposition inverse to the fundamental lemma (2), or (3), any formal identity containing three terms may be employed as the symbolical basis of finite expansions. In fact, if the identity is

$$\Xi_A = \Xi_1 A_1 + \Xi_2 A_2,$$

* Elliott, *Algebra of Quantics*, p. 115.

then functions $f = \Xi_A^m$ and corresponding expansions in the arguments A_i exist for all values of m. Gordan's* series may be developed (see § 6) from this point of view, from the formal identity

$$a_x b_y - a_y b_x = (ab)(xy).$$

This is also the principle upon which Hermite's theory of typical representation† and associated forms is based.

§ 2. Extension of the fundamental lemma.

The extension of (2) to the case of the general homogeneous expansion in p arguments can be readily made. We set

$$(4') \quad f = \Xi_A^m = (\Xi_1 A_1 + \Xi_2 A_2 + \cdots + \Xi_p A_p)^m = \Xi_1^m A_1^m + \cdots = \Phi_1 A_1^m + \cdots,$$

and assume that the property (A) is satisfied by all of the symbolical operators of the set

$$\nabla_i = \Xi_{i+1} \frac{\partial}{\partial \Xi_i} \qquad (i = 1, 2, \cdots, p-1).$$

Then it is easy to verify, by actually performing the indicated differentiations, that (4′) may be written

$$f = \sum \frac{\nabla_1^{i_1} \nabla_2^{i_2} \cdots \nabla_{p-1}^{i_{p-1}} \Xi_1^m}{\underline{|i_1}\,\underline{|i_2} \cdots \underline{|i_{p-1}}} A_1^{m-i_1} A_2^{i_1-i_2} \cdots A_p^{i_{p-1}},$$

where the set $(m - i_1, i_1 - i_2, \cdots, i_{p-2} - i_{p-1}, i_{p-1})$ ranges over the partitions of m into p parts, assuming each partition once and once only. Hence if, according to the property (A), ∇_i' is the non-symbolical equivalent of ∇_i, then there exists a function Φ and a set of $p-1$ operators ∇_i' such that the explicit expansion of f is (§ 5)

$$(4) \qquad f = \sum \frac{\nabla_1'^{i_1} \nabla_2'^{i_2} \cdots \nabla_{p-1}'^{i_{p-1}} \Phi}{\underline{|i_1}\,\underline{|i_2} \cdots \underline{|i_{p-1}}} A_1^{m-i_1} A_2^{i_1-i_2} \cdots A_p^{i_{p-1}}.$$

§ 3. On expressing one polynomial in terms of another.

Non-homogeneous variables will be used throughout this section. We let

$$f(x) = a_x^m = a_0 x^m + a_1 x^{m-1} + \cdots + a_m \quad (a_0 = 1, m = \mu\nu),$$

$$\varphi(x) = \xi_x^\nu = \xi_0 x^\nu + \xi_1 x^{\nu-1} + \cdots + \xi_\nu \equiv \xi_0 (x - s_1)(x - s_2) \cdots (x - s_\nu).$$

* Gordan, *Vorlesungen über Invariantentheorie*, vol. 2, § 117.

† Hermite, Journal für Mathematik, vol. 52 (1855), p. 23.

It may be shown by a form of the Euclidean process of successive division that there exists an expansion of f in powers of ξ_x^ν,

$$(5) \quad f(x) = \Phi_1 + \Phi_2\,\xi_x^\nu + \Phi_3\,(\xi_x^\nu)^2 + \cdots + \Phi_\mu\,(\xi_x^\nu)^{\mu-1} + \Phi_{\mu+1}\,(\xi_x^\nu)^\mu,$$

where Φ_i is a polynomial in x of order $\nu - 1$, and in particular, say

$$\Phi_1 = \gamma_0\,x^{\nu-1} + \gamma_1\,x^{\nu-2} + \cdots + \gamma_{\nu-1}.$$

We proceed to the determination of (the coefficients of) Φ_1 and the operator ∇ in (2). These will then give the whole expansion, and it will appear that (5) has a symbolical basis of the form

$$(6) \qquad \Xi_{(\phi)} = \Xi_{1f(x)} + \Xi_{2f(x)}\,\varphi.$$

Substitution of the roots of $\varphi(x)$ for x in (5) gives at once

$$(7) \qquad f(s_i) = \gamma_0\,s_i^{\nu-1} + \gamma_1\,s_i^{\nu-2} + \cdots + \gamma_{\nu-1} \quad (i = 1, 2, \cdots, \nu).$$

These are linear equations in the γ_j $(j = 0, 1, \cdots, \nu - 1)$. Their solution is possible provided $D \neq 0$, where D is the discriminant of $\varphi(x)$. This solution yields

$$(8) \qquad \gamma_j = \frac{\begin{vmatrix} s_1^{\nu-1} & s_1^{\nu-2} & \cdots & s_1^{\nu-j} & f(s_1) & s_1^{\nu-j-2} & \cdots & s_1 & 1 \\ s_2^{\nu-1} & s_2^{\nu-2} & \cdots & s_2^{\nu-j} & f(s_2) & s_2^{\nu-j-2} & \cdots & s_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_\nu^{\nu-1} & s_\nu^{\nu-2} & \cdots & s_\nu^{\nu-j} & f(s_\nu) & s_\nu^{\nu-j-2} & \cdots & s_\nu & 1 \end{vmatrix}}{\sqrt{D}}$$

$$(j = 0, 1, 2, \cdots, \nu - 1).$$

Thus γ_j is symmetric in the roots s_i, rational in the coefficients ξ_k $(k = 0, 1, \cdots, \nu)$ and linear in $a_0\,(=1), a_1, a_2, \cdots, a_m$. It is necessary to evaluate the symmetric functions γ_j of the roots s_i in order to determine the operator ∇ required by the theory of the symbolical basis, and (2). I did this originally by means of the Euclidean process mentioned above, i. e., $f(x)$ was divided by φ^μ, the remainder by $\varphi^{\mu-1}$, and so forth.* The results for the general case are given below, without details of derivation. In these formulas, Ω indicates the following operator:

$$\Omega = a_0\frac{\partial}{\partial a_1} + a_1\frac{\partial}{\partial a_2} + \cdots + a_{m-1}\frac{\partial}{\partial a_m}.$$

* This work was subsequently checked by inductive steps based upon ordinary computation by symmetric functions.

I obtain

$$
\begin{aligned}
F = \gamma_0 = {} & \Omega^{\nu-1} f(-\xi_1) + \sum_{i=2}^{k_{11}} \xi_i \, \Omega^{\nu+i-2} f'(-\xi_1) \\
& + \sum_{i=2}^{k_{21}} \sum_{j=2}^{k_{22}} \frac{\xi_i \, \xi_j}{\lfloor 2} \, \Omega^{\nu+i+j-3} f''(-\xi_1) \\
& + \sum_{i=2}^{k_{31}} \sum_{j=2}^{k_{32}} \sum_{k=2}^{k_{33}} \frac{\xi_i \, \xi_j \, \xi_k}{\lfloor 3} \, \Omega^{\nu+i+j+k-4} f'''(-\xi_1) \qquad (9) \\
& + \cdots \cdots \cdots \cdots \\
& + \sum_{i=2}^{k_{t1}} \sum_{j=2}^{k_{t2}} \cdots \sum_{n=2}^{k_{tn}} \frac{\xi_i \, \xi_j \cdots \xi_n}{\lfloor t} \, \Omega^{\nu+i+j+\cdots+n-t-1} f^{(t)}(-\xi_1) \\
& \qquad (k_{rs} \leqq \nu, \; k_{t1} + k_{t2} + \cdots + k_{tn} \leqq (\mu-1)\nu + 1).
\end{aligned}
$$

Then there results

$$
\begin{aligned}
\Phi = \Phi_1 = {} & F x^{\nu-1} + [-(\xi_2 \, \Omega + \xi_3 \, \Omega^2 + \cdots + \xi_\nu \, \Omega^{\nu-1}) F + a_{m-\nu+2}] \, x^{\nu-2} \\
& + [-(\xi_3 \, \Omega + \xi_4 \, \Omega^2 + \cdots + \xi_\nu \, \Omega^{\nu-2}) F + a_{m-\nu+3}] \, x^{\nu-3} \\
& + \cdots \cdots \cdots \cdots \qquad (10) \\
& + [-(\xi_{\nu-1} \, \Omega + \xi_\nu \, \Omega^2) F + a_{m-1}] \, x \\
& + [-\xi_\nu \, \Omega F + a_m].
\end{aligned}
$$

Furthermore, we find during the course of the derivation of these results by the division process that an operator ∇ in (2) exists and is precisely [see (6)]

$$
\nabla = \Xi_{2f(x)} \frac{\partial}{\partial \, \Xi_{1f(x)}} = -\frac{\partial}{\partial \xi_\nu}. \qquad (11)
$$

Hence expansion (5) takes the form

$$
\begin{aligned}
f(x) = {} & \Phi - \frac{\partial \Phi}{\partial \xi_\nu} \xi_x^\nu + \frac{\partial^2 \Phi}{\partial (\xi_\nu)^2} \frac{(\xi_x^\nu)^2}{\lfloor 2} - \cdots + (-1)^{\mu-1} \frac{\partial^{\mu-1} \Phi}{\partial (\xi_\nu)^{\mu-1}} \frac{(\xi_x^\nu)^{\mu-1}}{\lfloor \mu - 1} \\
& + (-1)^\mu \frac{\partial^\mu \Phi}{\partial (\xi_\nu)^\mu} \frac{(\xi_x^\nu)^\mu}{\lfloor \mu}.
\end{aligned} \qquad (12)
$$

It is thus analogous to the Taylor expansion, to which it reduces when $\nu = 1$.

We note in passing the following interesting property of the function F:

$$
-(\xi_1 \, \Omega + \xi_2 \, \Omega^2 + \cdots + \xi_\nu \, \Omega^\nu) F + a_{m-\nu+1} \equiv F.
$$

§ 4. Homogeneous expansions.

As an illustration of formula (3), we consider the expansion of a homogeneous binary form in terms of two other binary forms of lower order. Let

$$f_M = a_x^M = \prod_{i=1}^{M} (r_2^{(i)} x_1 - r_1^{(i)} x_2) = a_0 x_1^M + \cdots$$

be the form to be expanded. Let the arguments of the expansion be

$$A_1 = f_{1n} = \alpha_x^n = \prod_{j=1}^{n} (\alpha_2^{(j)} x_1 - \alpha_1^{(j)} x_2) = \alpha_0 x_1^n + \cdots,$$

$$A_2 = f_{2n} = \beta_x^n = \prod_{j=1}^{n} (\beta_2^{(j)} x_1 - \beta_1^{(j)} x_2) = \beta_0 x_1^n + \cdots.$$

Then the expansion sought will be of the type

$$f_M = \varphi_{0p} f_{1n}^m + \varphi_{1p} f_{1n}^{m-1} f_{2n} + \cdots + \varphi_{mp} f_{2n}^m, \tag{13}$$

where φ_{ip} is a homogeneous binary form of order p.

It is assumed at the outset that f_M, f_{in} are perfectly general forms, i. e., that their coefficients are independent variables. These forms are also understood to be given in the sense that the coefficient forms φ_{ip} are to be determined from them. Hence the totality of coefficients of the φ_{ip} ($i = 0, 1, \cdots, m$) must be equal in number to those of f_M,

$$M + 1 = (m + 1)(p + 1). \tag{14}$$

Also

$$M = mn + p.$$

Thus

$$p = n - 1, \qquad M = n(m + 1) - 1.$$

When M and p have these values the solution of $n(m + 1)$ linear equations will determine the coefficients of φ_{in-1} ($i = 0, 1, \cdots, m$), and consequently the whole expansion. Hence the expansion exists provided these linear equations are consistent. It will be shown that the determinant representing the condition for consistency is a power of the resultant of f_{1n}, f_{2n}, multiplied by a constant.

For this determinant D is of order $n(m + 1)$, and is of the form of the eliminant of the set of $m + 1$ forms of order mn,

$$f_{1n}^m, \quad f_{1n}^{m-1} f_{2n}, \quad \cdots, \quad f_{2n}^m.$$

If $D = 0$, then there exists a linear relation of the type

$$\lambda_1 f_{1n}^m + \lambda_2 f_{1n}^{m-1} f_{2n} + \cdots + \lambda_{m+1} f_{2n}^m = 0, \tag{15}$$

where λ_{m+1} is of order $n-1$ in (x), and so f_{1n}, f_{2n} have at least one common root. Conversely if f_{1n}, f_{2n} have a common root then $D=0$, for otherwise $f_{n(m+1)-1}$ would have a root of multiplicity m.

An expansion (13) *of* $f_{n(m+1)-1}$ *exists provided the resultant* $R(\alpha, \beta)$ *of* f_{1n}, f_{2n} *does not vanish.*

To determine our expansion (13), let

$$(16)\quad \varphi_{in-1} = p_{i0}\, x_1^{n-1} + p_{i1}\, x_1^{n-2} x_2 + \cdots + p_{in-1}\, x_2^{n-1} = p_{ix}^{n-1} \quad (i = 0, 1, \cdots m).$$

Substituting $\beta^{(k)}$ for x in (13) we get

$$(17)\qquad p_{0\beta^{(k)}}^{n-1} (\alpha_{\beta^{(k)}}^{n})^m = a_{\beta^{(k)}}^{n(m+1)-1} \qquad (k = 1, 2, \cdots n).$$

Also owing to the symmetry of (13) in (α, β),

$$(18)\qquad p_{m\alpha^{(k)}}^{n-1} (\beta_{\alpha^{(k)}}^{n})^m = a_{\alpha^{(k)}}^{n(m+1)-1} \qquad (k = 1, 2, \cdots, n).$$

Equations (17) form a set of n linear equations for the determination of the n coefficients of p_{0x}^{n-1}. In order to exhibit the solutions as briefly as may be let Δ_β be the discriminant of β_x^n and Δ_α that of α_x^n. Then

$$\pm \Delta_\beta = |\, \beta_1^{(1)n-1},\ \beta_1^{(2)n-2}\beta_2^{(2)},\ \beta_1^{(3)n-3}\beta_2^{(3)2},\ \cdots,\ \beta_2^{(n)n-1}\,|^2 \div \prod_{k=1}^{n} \beta_2^{(k)\beta(n-1)}.$$

Also let

$$\Sigma_j(\alpha, \beta, a) = |\, \alpha_{\beta^{(1)}}^{mn} \beta_1^{(1)n-1},\ \alpha_{\beta^{(2)}}^{mn} \beta_1^{(2)n-2}\beta_2^{(2)},\ \cdots,\ \alpha_{\beta^{(j)}}^{mn} \beta_1^{(j)n-j}\beta_2^{(j)j-1},$$

$$a_{\beta^{(j+1)}}^{n(m+1)-1},\ \alpha_{\beta^{(j+2)}}^{mn} \beta_1^{(j+2)n-j-2}\beta_2^{(j+2)j+2},\ \cdots,\ \alpha_{\beta^{(n)}}^{mn} \beta_2^{(n)n-1}\,|,$$

then we get

$$(19)\qquad p_{0j} = \frac{1}{R(\alpha, \beta)^m} \cdot \frac{\Sigma_j(\alpha, \beta, a)}{\prod_{k=1}^{n} \beta_2^{(k)n-1} \sqrt{\Delta_\beta}} \quad (j = 0, 1, \cdots, n-1).$$

From symmetry, or directly from (18), we have also

$$(20)\qquad p_{mj} = \frac{1}{R(\beta, \alpha)^m} \cdot \frac{\Sigma_j(\beta, \alpha, a)}{\prod_{k=1}^{n} \alpha^{(k)n-1} \sqrt{\Delta_\alpha}} \quad (j = 0, 1, \cdots, n-1).$$

The coefficients of $R(\alpha, \beta)^{-m}$ in (19), (20) are rational, *integral* and symmetric in the roots of f_{1n}, f_{2n} and hence rational in the quantities $\alpha_0, \alpha_1, \cdots; \beta_0, \beta_1, \cdots$.

Referring now to expansion formula (3) we note that the functions Φ_i $(i=1,2)$ are determined in the present case by (19), (20).

The operators ∇_1', ∇_2' required by (3) to complete the determination of (13)

are the Aronhold operators obtained from α_x^n, β_x^n; i. e.,

$$(21)\qquad \begin{aligned} -\nabla_1' &= \alpha_0\frac{\partial}{\partial\beta_0}+\alpha_1\frac{\partial}{\partial\beta_1}+\cdots+\alpha_n\frac{\partial}{\partial\beta_n},\\ -\nabla_2' &= \beta_0\frac{\partial}{\partial\alpha_0}+\beta_1\frac{\partial}{\partial\alpha_1}+\cdots+\beta_n\frac{\partial}{\partial\alpha_n}. \end{aligned}$$

These with $\Phi_1=\varphi_{0n-1}$, $\Phi_2=\varphi_{mn-1}$, $A_i=f_{in}$, give in explicit form the desired expansion of $f_{n(m+1)-1}$; that is, it takes the form (3):

$$(22)\qquad f_{n(m+1)-1}=\Phi_1 f_{1n}^m+\frac{1}{2}\left(\frac{\nabla_1'\Phi_1}{\underline{|1}}+\frac{\nabla_2'^{m-1}\Phi_2}{\underline{|m-1}}\right)f_{1n}^{m-1}f_{2n}+\cdots+\Phi_2 f_{2n}^m.$$

It is of some interest to enumerate the conditions, necessary and sufficient, in order that series (22), taken in non-homogeneous form with $x_2=1$, say, should have constant coefficients. We get a minimum set of such conditions by equating to zero all of the coefficients of φ_{in-1} $(i=0,1,\cdots,m)$ except the last.

Let (19) be written in the form

$$p_{0j}=\frac{Q_j(\alpha,\beta)}{R(\alpha,\beta)^m}\qquad (j=0,1,\cdots,n-1).$$

Then the aforesaid conditions are given by

$$(23)\qquad C_{ij}\equiv\nabla_1'^{i}Q_j=0\qquad \begin{pmatrix} i=0,1,\cdots,m\\ j=0,1,\cdots,n-2\end{pmatrix}.$$

Their number is

$$N=(n-1)(m+1).$$

It follows that a series free from conditions $(N=0)$, having constant coefficients, exists only when the arguments f_{in} are linear $(n=1)$.

By constructing equations analogous to (14) it is easy to show that a ternary form f_m can be expressed as a ternary expansion in three ternary arguments f_{in} $(i=1,2,3)$,

$$f_m=\varphi_{m00}f_{1n}^m+\cdots,$$

with the coefficient forms φ all of order $n-1$, and with the expansion free from conditions, only when the arguments are linear.

§ 5. Expansions with linear arguments.

Consider next the problem of expanding a p-ary form a_x^m in terms of powers of p linear p-ary forms α_{ix} $(i=1,2,\cdots,p)$. Expansion formula (4) applies, and

$$(24)\qquad a_x^m=\sum_{i_j=1}^{m}\frac{\nabla_1'^{i_1}\nabla_2'^{i_2}\cdots\nabla_{p-1}'^{i_{p-1}}\Phi_1}{\underline{|i_1}\,\underline{|i_2}\cdots\underline{|i_{p-1}}}\alpha_{1x}^{m-i_1}\alpha_{2x}^{i_1-i_2}\cdots\alpha_{px}^{i_{p-1}}.$$

In this the coefficient Φ_j is given by the set

$$i_1 = i_2 = \cdots = i_{j-1} = m; \qquad i_j = i_{j+1} = \cdots = i_{p-1} = 0 \quad (j = 1, 2, \cdots, p).$$

To get Φ_1 we substitute for (x) in (24) the coördinates of the point of intersection of the $p - 1$ hyperplanes

$$\alpha_{ix} = \alpha_{i1} x_1 + \alpha_{i2} x_2 + \cdots + \alpha_{ip} x_p = 0 \quad (i = 2, 3, \cdots, p)$$

This gives

$$x_j = - \frac{(\alpha_2 \alpha_3 \cdots \alpha_{j-1} \alpha_p \alpha_{j+1} \cdots \alpha_{p-1}) x_p}{(\alpha_2 \alpha_3 \cdots \alpha_p)} \quad (j = 1, 2, \cdots, p),$$

and by substitution

$$\Phi_1 = \frac{(a\alpha_2 \alpha_3 \cdots \alpha_p)^m}{(\alpha_1 \alpha_2 \cdots \alpha_p)^m}.$$

For this expansion there exists a symbolical basis in the form of (4'), as may readily be verified by considering (24) to be the transformed of a_x^m by the inverse of

$$X_i = \alpha_{ix} \qquad (i = 1, 2, \cdots, p).$$

The operators for this case are the Aronhold operators

$$-\nabla_i' = \alpha_{i1} \frac{\partial}{\partial \alpha_{i+11}} + \alpha_{i2} \frac{\partial}{\partial \alpha_{i+12}} + \cdots + \alpha_{ip} \frac{\partial}{\partial \alpha_{i+1p}} \quad (i = 1, 2, \cdots, p-1),$$

the whole expansion (24) being given by Φ_1 and ∇_i'.

It can be shown that corresponding results hold for forms whose coefficients belong to a field or reduced residue system* [modd p, $P(x)$], where p is a prime number and $P(x)$ an irreducible quantic modulo p of order n. Thus when $m = p = 2$, we have, provided $(\alpha\beta) \not\equiv 0$ [modd p, $P(x)$], an expansion of the form

$$f \equiv \Phi_1 (\alpha_1 x_1 - \alpha_2 x_2)^2 + I (\alpha_1 x_1 - \alpha_2 x_2)(\beta_1 x_1 - \beta_2 x_2) + \Phi_2 (\beta_1 x_1 - \beta_2 x_2)^2 \text{ [modd } p, P(x)],$$

where the linear arguments have coefficients belonging to the residue system. Then

$$\Phi_1 \equiv (\alpha_1 \beta_2 - \alpha_2 \beta_1)^{p^n-3} (a_0 \beta_2^2 + a_1 \beta_2 \beta_1 + a_2 \beta_1^2),$$

$$\Phi_2 \equiv (\alpha_1 \beta_2 - \alpha_2 \beta_1)^{p^n-3} (a_0 \alpha_2^2 + a_1 \alpha_2 \alpha_1 + a_2 \alpha_1^2) \text{ [modd } p, P(x)],$$

* Dickson, *Linear Groups*, p. 7.

and with

$$\nabla_1' = -\left(\alpha_1 \frac{\partial}{\partial \beta_1} + \alpha_2 \frac{\partial}{\partial \beta_2}\right), \qquad \nabla_2' = -\left(\beta_1 \frac{\partial}{\partial \alpha_1} + \beta_2 \frac{\partial}{\partial \alpha_2}\right),$$

we have

$$I = \tfrac{1}{2}(\Delta_1' \Phi_1 + \nabla_2' \Phi_2) \equiv -(\alpha\beta)^{n-3}[2a_0\alpha_2\beta_2 + a_1(\alpha_1\beta_2 + \alpha_2\beta_1) + 2a_2\alpha_1\beta_1] \qquad [\text{modd } p, P(x)].$$

The form $f = a_x b_x$ is apolar to $\varphi = (\alpha_1 x_1 - \alpha_2 x_2)(\beta_1 x_1 - \beta_2 x_2) = \alpha_x \beta_x$ provided $I \equiv 0$. In fact the latter congruence is a necessary and sufficient condition that the two forms f, φ respectively be the following linear combinations of two squares in the field:

$$f \equiv s\alpha_x^2 + t\beta_x^2; \qquad \varphi \equiv \sigma a_x^2 + \tau b_x^2.$$

§ 6. The Clebsch-Gordan Expansion.

Gordan's series* is a case of expansion (2). It may be of interest to give a sketch of the derivation of this series from the present point of view, for two reasons: first to show how the preceding general theory applies, and secondly because our methods give a very compact treatment of the details of the derivation, which is believed to be novel.

We employ as symbolical basis the formal identity

$$(E) \qquad a_x b_y - a_y b_x = (ab)(xy).$$

From it we have

$$a_x^m b_y^n \equiv [a_y b_x + (ab)(xy)]^m b_y^{n-m} \qquad (n \geqq m),$$

or

$$a_x^m b_y^n = a_y^m b_x^m b_y^{n-m} + \binom{m}{1} a_y^{m-1} b_x^{m-1} b_y^{n-m}(ab)(xy) + \cdots + \binom{m}{k} a_y^{m-k} b_x^{m-k} b_y^{n-m}(ab)^k (xy)^k + \cdots + (ab)^m (xy)^m b_y^{n-m}. \tag{25}$$

This is one form of Gordan's series. Here we have (§ 1)

$$\Phi = a_y^m b_x^m b_y^{n-m}, \qquad \nabla = \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}.$$

But to reduce the series to the ordinary form we use the following polar formula:

$$(a_x^m, b_x^n)_{y^{n-k}}^k = \frac{(ab)^k}{\binom{m+n-2k}{n-k}} \sum_{h=0}^{m-k} \binom{m-k}{m-k-h}\binom{n-k}{n-m+h} a_y^{m-k-h} a_x^h b_x^{m-k-h} b_y^{n-m+h}. \tag{26}$$

* Gordan, *Ueber Combinanten*, Mathematische Annalen, vol. 5 (1872), p. 95.

Since the sum of the numerical coefficients in the polar of a product is unity, (26) can be written

$$(a_x^m, b_x^n)^k_{y^{n-k}} = (ab)^k\, a_y^{m-k}\, b_x^{m-k}\, b_y^{n-m}$$

$$(27)\qquad + \frac{(ab)^k}{\binom{m+n-2k}{n-k}} \sum_{h=1}^{m-k} \binom{m-k}{m-k-h}\binom{n-k}{n-m+h} a_y^{m-k-h}\, b_x^{m-k-h}\, b_y^{n-m} \times (a_x^h\, b_y^h - a_y^h\, b_x^h).$$

The first term of this is, aside from the factor $\binom{m}{k}$, the coefficient of $(xy)^k$ in (25). Hence this coefficient is the $(n-k)$th transvectant of a_x^m and b_x^n, minus terms which contain the factor $(ab)^{k+1}(xy)$. Hence (27) may be used as a recursion formula; and, starting with $k = m$, we get

$$(ab)^m\, b_y^{n-m} = (a_x^m, b_x^n)^m_{y^{n-m}},$$

$$(ab)^{m-1}\, a_y\, b_x\, b_y^{n-m} = (a_x^m, b_x^n)^{m-1}_{y^{n-m+1}} - \frac{1}{n-m+2}(a_x^m, b_x^n)^m_{y^{n-m}}(xy),$$

$$(28)\qquad (ab)^{m-2} a_y^2 b_x^2 b_y^{n-m} = (a_x^m, b_x^n)^{m-2}_{y^{n-m+2}} - \frac{4}{n-m+4}(a_x^m, b_x^n)^{m-1}_{y^{n-m+1}}(xy) + \frac{2}{(n-m+2)(n-m+3)}(a_x^m, b_x^n)^m_{y^{n-m}}(xy)^2.$$

And, in general, by induction

$$(29)\qquad (ab)^{m-k}\, a_y^k\, b_x^k\, b_y^{n-m} = \alpha_0 (a_x^m, b_x^n)^{m-k}_{y^{n-m+k}} + \alpha_1 (a_x^m, b_x^n)^{m-k+1}_{y^{n-m+k-1}}(xy) + \cdots + \alpha_j (a_x^m, b_x^n)^{m-k+j}_{y^{n-m+k-j}}(xy)^j + \cdots + \alpha_k (a_x^m, b_x^n)^m_{y^{n-m}}(xy)^k.$$

Consequently it follows at once from (25) that

$$(30)\qquad a_x^m\, b_y^n = C_0 (a_x^m, b_x^n)^0_{y^n} + C_1 (a_x^m, b_x^n)^1_{y^{n-1}}(xy) + \cdots + C_j (a_x^m, b_x^n)^j_{y^{n-j}}(xy)^j + \cdots + C_m (a_x^m, b_x^n)^m_{y^{n-m}}(xy)^m$$

The constants C_i may now be determined in the usual manner by operating on both sides of (30) with the proper power of Δ, and noting that when $y \sim x$ in any polar it goes back into the original polarized form. The following known formula* may be used for this purpose:

$$(31)\qquad \Delta^i (xy)^j\, \alpha_x^{n-j}\, \alpha_y^{m-j} = \frac{\lfloor j \quad \lfloor m+n-j+1}{\lfloor j-i \quad \lfloor m+n-j-i+1}\, (xy)^{j-i}\, \alpha_x^{n-j}\, \alpha_y^{m-j} \qquad (i \leqq j).$$

* Grace and Young, *Algebra of Invariants*, p. 54.

The coefficients may also be derived by multiplying the respective members of the column (28) by

$$\binom{m}{m}(xy)^m,\ \binom{m}{m-1}(xy)^{m-1},\ \cdots,\ \binom{m}{2}(xy)^2,\ \binom{m}{1}(xy),\ 1;$$

and adding the results. Either method gives the well-known result

$$a_x^m b_y^n = \sum_{i=0}^{m} \frac{\binom{m}{i}\binom{n}{i}}{\binom{m+n-i+1}{i}} (xy)^i (a_x^m, b_x^n)_{y^{n-i}}^i . \tag{32}$$

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LEBESGUE INTEGRALS CONTAINING A PARAMETER, WITH APPLICATIONS*

BY

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1. **Introduction.**—In a recent memoir Lebesgue† has studied the following problem. Let $f(t)$ belong to one of a certain group of families of functions. What conditions should be imposed on another function $\phi(t, n)$ involving the parameter n, in order that the Lebesgue integral of the product of f and ϕ may exist and approach zero, as n becomes infinite, for all functions f of the family in question? I have studied elsewhere‡ the same problem when the functions involved are functions of several variables instead of one. The object of the present paper is to continue Lebesgue's work by considering other families of which most are narrower than his, and to make certain applications of the results. I have been led to do this because of an attempt to apply Lebesgue's theorems to certain series of orthogonal functions. It was found that the conditions were too severe, and the need was felt of simpler conditions, even at the expense of restricting the classes of functions for which the results would be valid. The conditions that should be imposed on ϕ in order that the integral of $f\phi$ shall remain limited (instead of approaching zero) are given in Theorem 4.

It is shown here how the theory may be applied to the series of Fourier and of Legendre to discover the degrees of their convergence and to find the rates at which their coefficients approach zero. In the case of the functions of Legendre certain inequalities are derived which may prove useful. The paper concludes with some theorems on the evaluation of definite integrals and on the integration of series.

2. The first theorem is taken from the paper of Lebesgue already mentioned. I copy it here, with certain slight changes, for which authority will be found in my earlier paper, because I shall need to refer to it frequently in the following pages.

* Presented to the Society April 27, 1912.

† Lebesgue, Annales de la Faculté de Toulouse, ser. 3, vol. 1 (1909), pp. 25–117.

‡ These Transactions, vol. 14 (1913), pp. 42–64.

THEOREM 1: *Necessary and sufficient conditions that the integral*

$$\int_k^l f(t)\,\phi(t,n)\,dt,$$

where f and ϕ are defined in the interval (k, l) and n is a parameter belonging to a sequence having plus infinity as its only limit point, shall exist for sufficiently large values of n as an absolutely convergent L-integral, and approach zero for all functions f belonging to one of the families specified below, may be stated in the several cases as follows:*

(F_1) In the case of the family F_1 of absolutely L-integrable functions, (1°) *M shall exist so that $|\phi|$ is less than M if $n > n_M$† at all points of (k, l), except perhaps at a null set, and* (2°) *the integral of ϕ over each interval (a, b) in (k, l) shall approach zero as n becomes infinite.*

(F_2) In the case of the family F_2 of functions whose squares are L-integrable, (1°) *M shall exist so that in (k, l) the integral of ϕ^2 is less than M if $n > n_M$, and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_3) In the case of the family F_3 of limited L-integrable functions, (1°) *the integral of ϕ shall be "equi-absolutely" continuous,‡ and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_4) In the case of the family F_4 of simply discontinuous§ functions, (1°) *M shall exist so that the integral of $|\phi|$ is less than M if $n > n_M$, and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_{4a}) In the case of the family F_{4a} of continuous functions, (1°) *condition* (1°) *for F_4 shall be satisfied,* (2°) *for each a in (k, l) the integral of $(t - a)\,\phi$ from a to l, and* (2a°) *the integral of ϕ from k to l shall approach zero as n becomes infinite.*

(F_5) In the case of the family F_5 of functions having limited variation, (1°) *ϕ shall be absolutely L-integrable,* (1a°) *M shall exist so that the integral of ϕ from k to t is less than M uniformly for all values of t if $n > n_M$, and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_{5a}) In the case of the family F_{5a} of continuous functions having limited variation, (1°, 1a°, 2°, 2a°) *conditions* (1°) *and* (1a°) *for F_5, and* (2°) *and* (2a°) *for F_{4a}, shall be satisfied.*

COROLLARY: *The conditions for F_1, F_2, and F_3 remain sufficient if, in place of* (2°), *it be required that $\phi(t, n)$ approach zero in (k, l), except perhaps at a null set of points.*

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* I. e., integral in the sense of Lebesgue.

† I use n_M in place of the usual n_0 to denote a fixed number.

‡ In the sense of Vitali, Rendiconti del Circolo Matematico di Palermo, vol. 23 (1907), p. 139. See also my other paper, loc. cit., p. 44.

§ I. e., having at most discontinuities of the first kind.

For I have shown in another paper* (by the use of a theorem of Vitali's) that termwise integration of the sequence $\{\phi(t, n)\}$ is permissible, if condition (1°) for the family in question is satisfied. Therefore, if ϕ approaches zero, so does its integral.

3. The following theorem is of interest because of its bearing on the development of an arbitrary function in series of orthogonal functions. There follow quite readily from it, for example, certain known facts concerning the convergence of Fourier's and Legendre's developments at the point x.

THEOREM 2: *In order that the integral of Theorem* 1 *shall exist for sufficiently large values of* n, *and approach zero for all functions* f *belonging to the family* F_3 *of functions which are continuous in* (k, l), *and have at a previously chosen point* x *a derivative, it is necessary and sufficient that* (1°) *the integral of* ϕ *from* k *to* l *approach zero as* n *becomes infinite, and that* (2°) *the conditions of Theorem* 1 *relative to* F_{4a} *be satisfied when* ϕ *is replaced by* $(t-x)\phi$.

Proof. Except for $t = x$, we have

$$f(t) = f(x) + (t-x)\frac{f(t)-f(x)}{t-x}$$

identically, and therefore

$$(1)\quad \int_k^l f\phi\,dt = f(x)\int_k^l \phi\,dt + \int_k^l \frac{f(t)-f(x)}{t-x}[(t-x)\phi]\,dt = \mathrm{I} + \mathrm{II}.$$

It is necessary and sufficient that (1) approach zero. I approaches zero if (1°) is fulfilled. Since, however, (1°) is clearly necessary (for if it is not satisfied the function $f = 1$ contradicts the theorem), it follows that it is necessary as well as sufficient that I and II separately approach zero. In II $[f(t)-f(x)]/(t-x)$ is, if properly defined at x, continuous, since $f(t)$ is continuous, and at x its derivative, that is, the limit of this fraction, exists. By (2°) we may therefore apply to II the conditions of Theorem 1 relative to F_{4a}, ϕ being replaced by $(t-x)\phi$, and it follows that II approaches zero, and the conditions stated are sufficient. Moreover the conditions in 2° are necessary, for, if any one of them is not satisfied, there exists by Theorem 1 a continuous function $q(t)$ such that

$$\int_k^l q(t)(t-x)\phi\,dt$$

does not approach zero. In that case the function $f = q(t)(t-x)$ contradicts the theorem, for it is continuous in (k, l) and has at x a derivative, viz:

$$\lim_{t=x}\frac{q(t)(t-x)-q(x)(x-x)}{t-x} = q(x).$$

* These Transactions, loc. cit., pp. 63, 64.

COROLLARY 1: *If condition* (1°) *be omitted, the theorem applies to the family of functions which have the properties of* F_6 *and, in addition, vanish at* x.

COROLLARY 2: *If, in the theorem, it is desired that the integral approach* $f(x)$ *instead of zero, it is necessary and sufficient that* (2°) *hold, and that in* (1°) *the integral be required to approach unity instead of zero.*

4. We now consider the same problem relative to the families G_1', $\cdots$, G_6'; G_1'', $\cdots$, G_6''; $\cdots$, defined as follows:

DEFINITION 1: The function f belongs to $G_j^{(i)}$ if

$$f(t) = C_i + \int_k^t f'\,dt,$$

where f' is a function belonging to $G_j^{(i-1)}$, and C_i is a constant, and $G_j^{(0)} = F_j$ (i. e., $f^{(i)}$ is in F_j). We do not here assert that f' is necessarily the derivative.

DEFINITION* 2: The function f belongs to $G_j^{(i)}$ if its $(i-1)$th derivative exists,† and is an indefinite integral, and its ith derivative (which will then exist except perhaps at a null set) may be defined, or redefined, in a null set so that it becomes a function of F_j.

Of these definitions the first will be the more convenient to use in our theoretical work, the second the easier to apply in practice. Before proceeding to discuss the problem, I will show that the two definitions are equivalent. The following theorems‡ will be used.

(*a*) *If* ϕ *is continuous in* (k, l), *it equals the derivative of its indefinite integral.*

(*b*) *The same is true, except perhaps at a null set, if* ϕ *is absolutely L-integrable in* (k, l).

(*c*) *If* f *is an indefinite integral, its derivative exists, except perhaps at a null set, and* f *is the indefinite integral of its derivative.*

(*d*) If df/dt *exists and is absolutely L-integrable in* (k, l), *then*

$$f(t) - f(k) = \int_k^t \frac{df}{dt}\,dt.$$

Assuming, first, Definition 1, let

$$f(t) = C_1 + \int_k^t f'\,dt, \tag{1}$$

* In particular, f satisfies this definition for $G_{4a}^{(i)}$, $G_{5a}^{(i)}$, or $G_6^{(i)}$, if its ith derivative is a member of F_{4a}, F_{5a}, or F_6, respectively. But a similar statement does not hold for $G_j^{(i)}$ when $j = 1, 2, 3,$ or 4. In fact, we cannot even say in these cases that, if df/dt exists, except perhaps at a null set, and after being defined, or redefined, in a null set becomes a member of F_j, then f belongs to $G_j^{(i)}$. For example, if $f = 0$ in $(0 \leqq t \leqq 1)$ and 1 in $(1 < t \leqq 2)$, df/dt exists in $(0, 2)$, except at 1, and if it be defined there in any manner it becomes a member of F_4, but f is not an integral.

† If a derivative is infinite at a point, I say it does not exist at that point.

‡ (*a*) is well known. For (*b*) and (*c*) see de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, 2d ed., vol. 1 (1909), p. 267; for (*d*) see page 263 of the same volume.

$f^{(i)}$ being in F_j. By (a), since by the definition f' is continuous when $i > 1$,

$$\frac{df}{dt} = f'(t) = C_2 + \int_k^t f''\,dt,$$

$$\frac{d^2f}{dt^2} = f''(t) = C_3 + \int_k^t f'''\,dt, \tag{2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^{i-1}f}{dt^{i-1}} = f^{(i-1)}(t) = C_i + \int_k^t f^{(i)}\,dt.$$

Then, since $f^{(i)}$ is given absolutely L-integrable, by (b), except perhaps at a null set,

$$\frac{d^if}{dt^i} = f^{(i)}(t). \tag{3}$$

We may define, or redefine, d^if/dt^i in this null set so that (3) is true at every point of (k, l). Then this new function belongs to the same family as $f^{(i)}$, namely to F_j.

Now let us suppose Definition 2 to be satisfied. Since $d^{i-1}f/dt^{i-1}$ is an indefinite integral, d^if/dt^i exists by (c), except perhaps at a null set, and, no matter how it is defined in this or in any other null set,

$$\frac{d^{i-1}f}{dt^{i-1}} - C_1 = \int_k^t \frac{d^if}{dt^i}\,dt.$$

By successive applications of (d), it follows that

$$\frac{d^{i-2}f}{dt^{i-2}} - C_2 = \int_k^t \frac{d^{i-1}f}{dt^{i-1}}\,dt,$$

etc. Finally,

$$f(t) - C_i = \int_k^t \frac{df}{dt}\,dt,$$

and df/dt belongs to $G_j^{(i-1)}$, since d^if/dt^i may be defined, or redefined, in a null set so that it belongs to F_j. This statement means that f satisfies the requirements of Definition 1.

We have now the following general theorem, which holds for all positive integral values of i.

THEOREM 3: *In order that the integral of Theorem* 1 *shall exist, for sufficiently large values of* n, *and approach zero as* n *becomes infinite, for all functions belonging to the family* * $G_j^{(i)}$, *it is necessary and sufficient that* (1°) *the conditions of the corresponding theorem for the family* $G_j^{(i-1)}$ *be satisfied* ($G_j^{(0)}$ *being* F_j) *when*

* Defined at the beginning of this section.

ϕ is replaced by its integral over (k, t), and that* (2°) *the integral of ϕ over (k, l) approach zero as n becomes infinite.*

Proof. Using Definition 1 and integrating by parts,† we have

$$(1) \qquad \int_k^l f\phi\,dt = f(l)\int_k^l \phi\,dt - \int_k^l \left(\int_k^t \phi\,dt\right) f'\,dt = \mathrm{I} + \mathrm{II}.$$

By (2°), I approaches zero. By (1°) and the corresponding theorem for $G_j^{(i-1)}$, II approaches zero. Hence the conditions are *sufficient.* To show that they are necessary we note first that the function $f = 1$ proves that (2°) is necessary, and that therefore, in (1), I must approach zero. It follows that II must approach zero. Now, if (1°) is not satisfied, there exists, by the theorem for $G_j^{(i-1)}$, a function $q(t)$ belonging to $G_j^{(i-1)}$ such that

$$(2) \qquad \int_k^l \left(\int_k^t \phi\,dt\right) q\,dt$$

does not approach zero. Let $f = \int_k^t q\,dt$. Then f belongs to $G_j^{(i)}$ and (1) holds, q replacing f'; but by (2) II does not approach zero, a condition which we have just found to be necessary.

COROLLARY 1: *If condition* (2°) *be omitted, the theorem applies to that family of functions which have the properties of $G_j^{(i)}$ $(i > 0)$ and, in addition, vanish at l.*

Sufficiency.—In (1), II approaches zero as before, and I vanishes since $f(l) = 0$.

Necessity.—Suppose (1°) to be not satisfied, and consider the function $q(t)$ given in (2). Let

$$f(t) = \int_k^t q(t)\,dt - \int_k^t q(t)\,dt.$$

For this function we may write, analogous to (1),

$$\int_k^l f\phi\,dt = \left[f(t)\int_k^t \phi\,dt\right]_k^l - \int_k^l \left(\int_k^t \phi\,dt\right) q\,dt = -\int_k^l \left(\int_k^t \phi\,dt\right) q\,dt,$$

and, by (2), this does not approach zero.

COROLLARY 2: *In order that the integral from k to k' of $f\phi$ shall approach $f(l)$ for all functions f which belong to $G_j^{(i)}$ $(i > 0,\ j < 6)$ in the interval $(k \lesseqgtr l \lesseqgtr k')$, it is sufficient that the conditions of Corollary* 1 *be satisfied for each of the intervals (k, l), (k', l), and that the integral from k to k' of ϕ approach unity.*

* This supposes that ϕ is absolutely L-integrable for sufficiently large values of n, a condition which is necessary, for if it is not satisfied the function $f = 1$ contradicts the theorem.

† Lebesgue, loc. cit., p. 46.

For, by the last condition the expressions

$$f(l) - \int_k^{k'} f\phi\, dt, \qquad \int_k^{k'} [f(l) - f(t)]\,\phi\, dt$$

have the same limit, and the function $f(l) - f(t)$ has the properties of $G_j^{(i)}$ in each of the intervals stated, and in addition vanishes at l. By the preceding corollary, therefore, the first expression approaches zero.

5. Sometimes it is desirable to know whether the integral of Theorem 1 remains limited,* not whether it approaches zero, as n becomes infinite. The following statement will enable one to determine from the previous theorems the necessary and sufficient conditions that this shall be true.

THEOREM 4: *If n is as in the preceding theorems and I_n is a number depending on it, then a necessary and sufficient condition that two numbers, M and n_M, exist so that I_n shall be numerically less than M when $n > n_M$ is that, for every positive divergent sequence $\{q_n\}$, $\lim_{n=\infty} | I_n | / q_n = 0$.*

APPLICATIONS.

6. **Fourier's Coefficients.**—In this section I discuss the rates at which the coefficients of Fourier approach zero.† Let f be defined and absolutely L-integrable in the interval $(c, c + 2\pi)$. As is customary, we write

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(t) \sin nt\, dt, \qquad b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(t) \cos nt\, dt.$$

Let $\{q_n\}$ be a positive sequence which diverges at an arbitrarily small, positive rate. From the preceding theorems the following may be easily established:

(*a*) *The coefficients a_n and b_n approach zero.*‡

(*b*) *There exist continuous, odd* § *functions for which $q_n a_n$ does not approach zero, and continuous, even functions for which $q_n b_n$ does not approach zero.*

(*c*) *If f has limited variation, na_n and nb_n remain finite.*‖

(*d*) *If f is continuous and has limited variation, but not necessarily if f has limited variation and is not continuous, na_n approches zero when c is an odd multiple of $\pi/2$ and n is odd, and nb_n approaches zero when c is an even multiple of $\pi/2$.*

(*e*)¶ *Let f belong to $G_1^{(i)}$, let $f^{(0)} = f$, $f^{(p)}$ be the pth derivative of f, and let $p = 0$,*

* Cf., e. g., § 6, *c*, *f*, and § 7, *c*, *f*.

† Cf. also a footnote to an article by Bôcher, Annals of Mathematics, ser. 2, vol. 7 (1906), p. 109.

‡ This follows from Theorem 1 for F_1, and is well known.

§ This shows that (*a*) is the most that may be said even of families of functions as restricted as those in (*b*). The function $f(t)$ is odd in $(c, c + 2\pi)$, if $f(c + \pi - t) = -f(c + \pi + t)$; even, if $f(c + \pi - t) = f(c + \pi + t)$.

‖ This is known.

¶ Cf. Young, Proceedings of the London Mathematical Society, ser. 2, vol. 10 (1911), pp. 256–7.

$1, \cdots, i-1$. *Then* $n^i a_n$ *approaches zero if* $c = 0$ *or a multiple of* π, *and* $f^{(p)}(c+2\pi) = f^{(p)}(c)$ *when* p *is even;* $n^i b_n$ *approaches zero if* $c = 0$ *or a multiple of* π, *and* $f^{(p)}(c+2\pi) = f^{(p)}(c)$ *when* p *is odd; both approach zero if* $f^{(p)}(c+2\pi) = f^{(p)}(c)$ *for all the values of* p.

(*f*) *If* f *belongs to* $G_5^{(i)}$, *then with the same provisions, respectively, as in* (*e*), $n^{i+1} a_n$ *and* $n^{i+1} b_n$ *remain finite.*

Proof of (*b*).—Consider condition 1° of Theorem 1 for F_{4a}.

$$\int_c^{c+\pi} q_n \left| \begin{matrix} \sin nt \\ \cos nt \end{matrix} \right| dt \gtrless q_n \int_c^{c+\pi} \begin{matrix} \sin^2 nt \\ \cos^2 nt \end{matrix} dt = \frac{\pi}{2} q_n,$$

which is not limited.

Proof of (*c*).—Consider Theorem 1 for F_5.

$$\frac{n}{q_n} \left| \int_c^{c+t} \begin{matrix} \sin nt \\ \cos nt \end{matrix} dt \right| = \frac{1}{q_n} \left| \begin{matrix} \cos nc - \cos n(c+t) \\ \sin n(c+t) - \sin nc \end{matrix} \right| \lessgtr \frac{2}{q_n},$$

which approaches zero.

Proof of (*d*).—Consider Theorem 1 for F_{5a}.

$$n \left| \int_c^{c+t} \begin{matrix} \sin nt \\ \cos nt \end{matrix} dt \right| \lessgtr 2.$$

$$n \left| \int_a^{c+2\pi} (t-a) \sin nt \, dt \right| = \frac{1}{n} \left| \sin nc - \sin na + n(a-c-2\pi) \cos nc \right|,$$

which approaches zero if nc is an odd multiple of $\pi/2$. Similarly,

$$n \left| \int_a^{c+2\pi} (t-a) \cos nt \, dt \right| = \frac{1}{n} \left| \cos nc - \cos na + n(c+2\pi-a) \sin nc \right|,$$

which approaches zero if c is an even multiple of $\pi/2$. The conditions of the theorems for F_{5a} are thus satisfied, but condition (2°) for F_5 is not satisfied. For example, if $c = 0$ and $t = \pi/2$,

$$n \left| \int_0^{\frac{\pi}{2}} \cos nt \, dt \right| = \left| \sin \frac{n\pi}{2} \right|,$$

which does not approach zero.

Proof of (*e*) and (*f*).—I use here a well-known method.* Integration by parts is allowable† because f is an indefinite integral, and so we have

$$\int_c^{c+2\pi} f \sin nt \, dt = -\frac{\cos nc}{n} [f(c+2\pi) - f(c)] + \frac{1}{n} \int_c^{c+2\pi} f' \cos nt \, dt.$$

* E. g., Hobson, *Theory of Functions of a Real Variable* (1907), p. 718.

† Lebesgue, loc. cit., p. 46.

Therefore, if (1) either $\cos nc$ or $f(c+2\pi)-f(c)$ equals zero,

$$a_n = \frac{b_n'}{n}, \qquad \text{where} \qquad b_n' = \frac{1}{\pi}\int_c^{c+2\pi} f' \cos nt\, dt.$$

Likewise

$$\int_c^{c+2\pi} f \cos nt\, dt = \frac{\sin nc}{n}\left[f(c+2\pi)-f(c)\right] - \frac{1}{n}\int_c^{c+2\pi} f' \sin nt\, dt,$$

and, if (2) either $\sin nc$ or $f(c+2\pi)-f(c)$ equals zero,

$$b_n = \frac{a_n'}{n}, \qquad \text{where} \qquad a_n' = \frac{1}{\pi}\int_c^{c+2\pi} f' \sin nt\, dt.$$

We can obviously satisfy (2) by choosing $c = 0$ or a multiple of π. We can satisfy both by making $f(c+2\pi) = f(c)$.

Repeating these processes and applying (a) we have (e); applying (c) we have (f).

From the foregoing propositions follow many corollaries concerning the convergence of Fourier's series. I will state two of them:

COROLLARY 1: *If f has limited variation, the series whose general term is $|a_n^p \sin^p nx| + |b_n^p \cos^p nx|$ converges uniformly when $p > 1$.* For, by (c), this may be compared with the series whose general term is $1/n^p$.

COROLLARY 2: *If f is the integral of a function of limited variation and if $f(c) = f(c+2\pi)$, the series of Corollary 1 converges uniformly when $p > 1/2$.* For, by (f) for G_5', it may be compared with the series whose general term is $1/n^{2p}$.

7. **Fourier's Development.**—In this section n and q_n are used as in the preceding section, and R_n represents the difference between $f(x)$ and the sum of the first terms of its Fourier's series up to and including the terms whose argument is nx. The function $f(t)$ is supposed defined everywhere, to be periodic of period 2π, and absolutely L-integrable in the interval $(-\pi, \pi)$. At x the limits $f(x+0)$ and $f(x-0)$ are supposed to exist, and $f(x)$ to be equal to one half their sum. As is well known,*

$$(1) \qquad \pi R_n = \int_{x-\pi}^{x+\pi} \left[f(t)-f(x)\right] \frac{\sin \dfrac{2n+1}{2}(t-x)}{2\sin \dfrac{t-x}{2}}\, dt$$

$$(2) \qquad = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[f(x+2v)-f(x)\right] \frac{\sin \nu v}{\sin v}\, dv \qquad (\nu = 2n+1).$$

Set

$$\psi(v) = \frac{f(x+2v)-f(x)}{\sin v} \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

* E. g., Pierpont, *Theory of Functions of Real Variables*, vol. 2, pp. 421, 2.

Reasoning analogous to that of the preceding article gives us the following propositions:

(*a*) *For some function* ψ *which is absolutely L-integrable,* $q_n R_n$ *does not approach zero, though it is known that* R_n *does approach zero.*

(*b*) *For some function* $f(t)$ *which is continuous in* $(-\pi, \pi)$ *and has a first derivative at* x, *the same is true.*

(*c*) *If* ψ *has limited variation,* nR_n *remains finite.*

(*d*) *The same is true if* $1/v[f(x+2v) - f(x)]$ *has limited variation in* $(-\pi/2, \pi/2)$.

(*e*) *If* ψ *is continuous and has limited variation,* nR_n *approaches zero, but this is not true of all functions having limited variation.*

(*f*) *If* ψ *is the integral of a function having limited variation (i. e., if* ψ *belongs to* G_5'), $n^2 R_n$ *remains finite.*

(*g*) *There exists a function* ψ *belonging to all the other G families, for which* $n^2 R_n$ *does not approach zero.*

Proof of (*a*).—We use (2) and condition 1° of Theorem 1 for F_1, putting $\phi = q_n \sin \nu v$. This function is not uniformly limited in $(-\pi/2, \pi/2)$. Therefore there exists an absolutely L-integrable function $g(v)$, such that the integral from $-\pi/2$ to $\pi/2$ of $g \sin \nu v$ does not approach zero. It can now be shown that a function $f(t)$ may be defined so that $g(v)$ has the form $\psi(v)$ above.

Proof of (*b*).—We use (1). The function $f(t) - f(x)$ satisfies the conditions on f in Corollary 1 of Theorem 2 for F_{6a}. We have, then, corresponding to ϕ,

$$\frac{q_n \sin \frac{\nu}{2}(t-x)}{2 \sin \frac{1}{2}(t-x)}.$$

Since, in the interval $(x-\pi, x+\pi)$, except at most at the three points, $x-\pi$, x, $x+\pi$,

$$\left| \frac{t-x}{2 \sin \frac{1}{2}(t-x)} \right| \leqq 1,$$

$$\int_{x-\pi}^{x+\pi} |(t-x)\phi| \, dt \leqq q_n \int_{x-\pi}^{x+\pi} \left| \sin \frac{\nu}{2}(t-x) \right| dt$$

$$\leqq q_n \int_{x-\pi}^{x+\pi} \sin^2 \frac{\nu}{2}(t-x) \, dt = q_n \pi,$$

and is not uniformly limited.

Proof of (*c*).—Using (2) and Theorem 1 for F_5, we have

$$\left|\frac{n}{q_n}\int_{-\frac{\pi}{2}}^{t}\sin \nu v\,dv\right| = \frac{n}{q_n \nu}\left|-\cos \nu t\right| \leqq \frac{n}{q_n \nu},$$

and this approaches zero.

Proof of (*d*).—We use the same theorem, but now corresponding to ϕ we have $v \sin \nu v / \sin v$. By an inequality,* due to Lebesgue, for monotone increasing functions, and since in $(0, \pi/2)$ $v/\sin v$ is monotone increasing from 1 to $\pi/2$, we have,† if $0 \leqq t \leqq \pi/2$,

$$\left|\int_0^t \frac{v \sin \nu v}{\sin v}\,dv\right| < \frac{3\pi}{2}\max_{\lambda \text{ in } (0,\,t)}\left|\int_0^{\lambda}\sin \nu v\,dv\right| \leqq \frac{3\pi}{2}\frac{2}{\nu}.$$

The same is true if $-\pi/2 \leqq t \leqq 0$, since the integrand is an odd function. Therefore

$$\frac{n}{q_n}\left|\int_{-\frac{\pi}{2}}^{t}\frac{v \sin \nu v}{\sin v}\,dv\right| < \frac{n}{q_n}\frac{6\pi}{\nu},$$

which approaches zero as n becomes infinite.

Proof of (*e*).—Using (2) and Theorem 1 for F_{5a}, we have

$$n\left|\int_{-\frac{\pi}{2}}^{t}\sin \nu v\,dv\right| \leqq \frac{2n}{\nu} \leqq 1,$$

$$n\left|\int_{a}^{\frac{\pi}{2}}(v-a)\sin \nu v\,dv\right| = \frac{n}{\nu^2}\left|\sin\frac{\nu\pi}{2} - \sin \nu a\right| \leqq \frac{2n}{\nu^2},$$

$$n\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin \nu v\,dv = 0.$$

These show that the conditions in question are satisfied, and establish the first part of (*e*). As for the second part, we note that

$$n\left|\int_{-\frac{\pi}{2}}^{t}\sin \nu v\,dv\right| = \frac{n}{\nu}\left|\cos \nu t\right|.$$

This equals n/ν when $t = 0$, and therefore does not approach zero for each t. We may now proceed as in the proof of (*a*).

* Lebesgue, loc. cit., p. 36, B. See also my paper (second footnote), p. 54.

† "Max" means the maximum of the function with respect to the parameter λ. I use the words "maximum" and "minimum" in the sense of upper and lower limits, not of greatest and least values.

Proof of (*f*).—Using (2) and Theorem 3 for G_5', we have

$$\frac{n^2}{q_n}\left|\int_{-\frac{\pi}{2}}^{t} dt \int_{-\frac{\pi}{2}}^{t} \sin \nu v \, dv\right| = \frac{n^2}{q_n \nu^2} |\sin \nu t \pm 1| \leqq \frac{2n^2}{q_n \nu^2},$$

$$\frac{n^2}{q_n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \nu v \, dv = 0.$$

Proof of (*g*).—It would be practicable to show that for no G are the conditions of Theorem 3 satisfied when $\phi = n^2 \sin \nu v$, but it is easier to see directly that for the function $\psi(v) = v$, which belongs to all the families, $n^2 |R_n|$ remains greater than $2/9$ if $n > 1$. $\psi = v$, if $f(t) = (t/2) \sin (t/2)$ in $(-\pi, \pi)$ and $x = 0$.

$$n^2 |R_n| = n^2 \left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v \sin \nu v \, dv\right| = \frac{2n^2}{\nu^2} > \frac{2}{9}, \quad \text{if } n > 1.$$

8. **Legendre's Development.**—In this section n and q_n are used as hitherto, $f(t)$ is supposed defined and absolutely L-integrable in the interval $(-1, 1)$, P_n is Legendre's polynomial of order n, and A_n is the coefficient of P_n in Legendre's development of f:

$$A_n = \frac{2n+1}{2}\int_{-1}^{1} f P_n \, dt.$$

In the following lemmas, M is some fixed number independent of t and t_i.

LEMMA 1:

(*a*) $$\left|\int_{-1}^{t} P_n \, dt\right| < \frac{M}{n^{\frac{1}{2}}(1-t^2)^{\frac{1}{4}}} \qquad (t^2 + 1,\ n \geqq 2),$$

(*b*) $$\left|\int_{-1}^{t} P_n dt\right| < \frac{M}{n^{1.1}} \qquad (n \geqq 2).$$

LEMMA 2: *If* $P_n(t)$ *be integrated* i *times, with respect to the variables* $t_0, t_1, \cdots t_{i-1}$,

(*a*) $$\left|\int_{-1}^{t_i} \cdots \int_{-1}^{t_1} P_n \, dt_0\right| < \frac{M}{n^{i+\frac{1}{2}}(1-t_i^2)^{\frac{1}{4}}} \qquad (t_i^2 + 1,\ n \geqq 2i),$$

(*b*) $$\left|\int_{-1}^{t_i} \cdots \int_{-1}^{t_1} P_n \, dt_0\right| < \frac{M}{n^{i+0.1}} \qquad (n \geqq 2i).$$

LEMMA* 3:

(*a*) $$\int_{-1}^{t} P_n^2 \, dt < \frac{1}{n} \qquad (n \geqq 1),$$

(*b*) $$\int_{-1}^{t_i}\left(\int_{-1}^{t_{i-1}} \cdots \int_{-1}^{t_1} P_n \, dt_0\right)^2 dt_{i-1} < \frac{M}{n^{2i-1}} \qquad (n \geqq 2i-2).$$

* Proposition (*a*) is known.

LEMMA* 4

$$(a) \qquad \int_{-1}^{t} |P_n|\,dt < \frac{M}{\sqrt{n}} \qquad (n \geqq 1),$$

$$(b) \qquad \int_{-1}^{t_i}\left|\int_{-1}^{t_{i-1}} \cdots \int_{-1}^{t_1} P_n\,dt_0\right| dt_{i-1} < \frac{M}{n^{i-\frac{1}{2}}} \qquad (n \geqq 2i-1).$$

Proof of Lemma 1.—From the known equation

$$(2n+1)\,P_n = \frac{d}{dt}(P_{n+1}) - \frac{d}{dt}(P_{n-1}),$$

we derive

$$(1) \qquad \left|\int_{-1}^{t} P_n\,dt\right| = \left|\int_{-1}^{1}\right| + \left|\int_{1}^{t}\right| = \frac{1}{2n+1}\,|P_{n+1}(t) - P_{n-1}(t)| \quad (n>0).$$

Hobson† has proved the inequality

$$(2) \qquad (1-t^2)^{\frac{1}{4}}\,|P_n(t)| < \frac{M}{\sqrt{n}} \qquad (n>0).$$

This with (1) proves that

$$\left|\int_{-1}^{t} P_n\,dt\right| < \frac{M}{2n+1}\left(\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n-1}}\right)\frac{1}{(1-t^2)^{\frac{1}{4}}} < \frac{3M}{n^{\frac{3}{2}}(1-t^2)^{\frac{1}{4}}}$$

$$(t^2+1,\ n \geqq 2),$$

which is (a). To establish (b) we note first that‡

$$(3) \qquad \left|\int_{-1}^{-1+x} \frac{dt}{(1-t^2)^{\frac{1}{4}}}\right|, \quad \left|\int_{1-x}^{1} \frac{dt}{(1-t^2)^{\frac{1}{4}}}\right| \leqq \frac{4}{3}x^{\frac{3}{4}} \qquad (0 \leqq x \leqq 1).$$

Case 1: $-1 + 1/n^{0.8} \leqq t \leqq 1 - 1/n^{0.8}$.

$$(4) \qquad \left|\int_{-1}^{t} P_n\,dt\right| \leqq \int_{-1}^{-1+1/n^{0.8}} |P_n|\,dt + \left|\int_{-1+1/n^{0.8}}^{t} P_n\,dt\right| = \mathrm{I} + \mathrm{II}.$$

By (2) and (3),

$$(5) \qquad \mathrm{I} < \frac{4}{3}\left(\frac{1}{n^{0.8}}\right)^{\frac{3}{4}}\frac{M}{\sqrt{n}} = \frac{4M}{3n^{1.1}} \qquad (n>0).$$

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* I do not need this, but I put it down as a simple corollary of the others; (a) is known.

† Hobson, Proceedings of the London Mathematical Society, ser. 2, vol. 7 (1909), p. 25.

‡ For example, the first of these equals, if we put $t = z - 1$,

$$\left|\int_0^x \frac{dz}{(2z-z^2)^{\frac{1}{4}}}\right| \leqq \left|\int_0^x \frac{dz}{z^{\frac{1}{4}}}\right| = \frac{4}{3}x^{\frac{3}{4}}, \text{ if } 0 \leqq x \leqq 1.$$

By (2), since $1 - t^2 \gtreqless 1/n^{1.6}$, and

$$\int_a^b P_n\,dt = \frac{1}{2n+1}\,[P_{n+1}(t) - P_{n-1}(t)]_a^b,$$

$$\text{(6)} \qquad \text{II} < \frac{1}{2n+1}\cdot\frac{4M}{\sqrt{n-1}}\cdot n^{0.4} < \frac{4M}{2n}\cdot\sqrt{\frac{2}{n}}\cdot n^{0.4} < \frac{4M}{n^{1.1}} \qquad (n>1).$$

The inequalities (5) and (6) establish Case 1.

Case 2: $-1 \lesseqgtr t \lesseqgtr -1 + 1/n^{0.8}$.

$$\left|\int_{-1}^{t} P_n dt\right| \lesseqgtr \int_{-1}^{-1+1/n^{0.8}} |P_n|\,dt,$$

and by (5) this is less than $M/n^{1.1}$ if $n > 0$. Likewise, if

Case 3: $1 - 1/n^{0.8} \lesseqgtr t \lesseqgtr 1$,

$$\left|\int_{-1}^{t} P_n dt\right| \lesseqgtr \left|\int_{-1}^{1-1/n^{0.8}} P_n dt\right| + \int_{1-1/n^{0.8}}^{t} |P_n|\,dt < \frac{2M}{n^{1.1}} \qquad (n>0),$$

by Case 1, (2), and (3).

Proof of Lemma 2.—Lemma 2 is proved by means of Lemma 1 and equalities like the following:

$$\int_{-1}^{t_2} dt_1 \int_{-1}^{t_1} P_n\,dt_0 = \frac{1}{2n+1}\left[\frac{1}{2n+3}P[_{n+2}(t_2) - P_n(t_2)]\right.$$
$$\left. - \frac{1}{2n-1}[P_n(t_2) - P_{n-2}(t_2)]\right],$$

$$\left|\int_{-1}^{t_2} dt_1 \int_{-1}^{t_1} P_n\,dt_0\right| = \left|\int_{-1}^{t_2} \frac{1}{2n+1}[P_{n+1}(t_1) - P_{n-1}(t_1)]\,dt_1\right|$$
$$\lesseqgtr \frac{1}{2n+1}\left(\left|\int_{-1}^{t_2} P_{n+1}\,dt_1\right| + \left|\int_{-1}^{t_2} P_{n-1}\,dt_1\right|\right).$$

Proof of Lemma 3(*b*).—By (*a*) of Lemma 2,

$$\int_{-1}^{t_i} (\qquad)^2 \cdot dt_{i-1} < \int_{-1}^{1} \frac{M^2}{n^{2i-1}\sqrt{(1-t_{i-1}^2)}}\,dt_{i-1} = \frac{\pi M^2}{n^{2i-1}} \qquad (n \gtreqless 2i-2).$$

Proof of Lemma 4(*b*).—By (*a*) of Lemma 2, and (3),

$$\int_{-1}^{t_i} |\qquad|\,dt_{i-1} < \int_{-1}^{1} \frac{M}{n^{i-\frac{1}{2}}(1-t_{i-1}^2)^{\frac{1}{2}}}\,dt_{i-1} = \int_{-1}^{0} + \int_{0}^{1} < \frac{2M}{n^{i-\frac{1}{2}}}\,\frac{4}{3} < \frac{4M}{n^{i-\frac{1}{2}}}$$
$$(n \gtreqless 2i-2).$$

The following statements are now easily established.

(*a*) *For every absolutely L-integrable function, A_n/n approaches zero, but there exists such a function for which $q_n A_n/n$ does not approach zero.*

(*b*) *For every function whose square is L-integrable,* $A_n/\sqrt{n}$ *approaches zero, but there exists such a function for which* $q_n A_n/\sqrt{n}$ *does not approach zero.*

(*c*) *There exists a continuous* function for which* $q_n A_n$ *does not approach zero.*

(*d*) *For every function of limited variation,* $n^{0.1} A_n$ *approaches zero.*

(*e*) *For every function of* $G_2^{(i)}$, $n^{i-\frac{1}{2}} A_n$ *approaches zero.*

(*f*) *For every function of* $G_5^{(i)}$, $n^{i+0.1} A_n$ *approaches zero.*

Proof of (*a*).—The first part of (*a*) follows from Theorem 1 for F_1, ϕ being† P_n, for P_n is limited, and by Lemma 1 (*b*) its integral from -1 to t approaches zero. But if ϕ be $q_n P_n$, condition (1°) of that theorem is not satisfied.

Proof of (*b*).—Using Theorem 1 for F_2, $\phi = \sqrt{n} P_n$, and Lemma 1 (*b*), we have

$$\int_{-1}^{1} \frac{n^2 P_n^2}{n}\, dt = \frac{2n}{2n+1} < 1 \qquad (n \geqq 1),$$

$$\int_{-1}^{t} \frac{n P_n}{\sqrt{n}}\, dt < \frac{M}{n^{0.6}}.$$

Proof of (*c*).—Condition (1°) of Theorem 1 for F_{4a} is not satisfied by $\phi = q_n n P_n$, for

$$q_n \int_{-1}^{1} n\,|P_n|\, dt \geqq q_n n \int_{-1}^{1} P_n^2\, dt = \frac{2n q_n}{2n+1}.$$

Proof of (*d*).—Putting $\phi = n^{1.1} P_n$ in Theorem 1 for F_5, we have, by Lemma 1 (*b*),

$$\left| \int_{-1}^{t} n^{1.1} P_n\, dt \right| < M \qquad (n \geqq 2),$$

and, by Lemma 1 (*a*),

$$\left| \int_{-1}^{t} n^{1.1} P_n\, dt \right| < \frac{n^{1.1} M}{n^{1.5} (1-t^2)^{\frac{1}{4}}} \quad \text{if } t^2 \neq 1 \qquad (n \geqq 2),$$

$$= 0 \quad \text{if } t^2 = 1 \qquad (n \geqq 1).$$

Proof of (*e*).—Consider G_2'. (The reasoning is analogous for the other G_2's.) Applying the first condition of Theorem 3 for G_2', and using $\phi = n^{\frac{3}{2}} P_n$, we have, by Lemma 3 (*b*), and Lemma 2 (*b*),

$$\int_{-1}^{1} \left(\int_{-1}^{t_1} n^{\frac{3}{2}} P_n\, dt_0 \right)^2 dt_1 < \frac{n^3 M}{n^3} = M \qquad (n \geqq 2),$$

$$\int_{-1}^{t_2} \left(\int_{-1}^{t_1} n^{\frac{3}{2}} P_n\, dt_0 \right) dt_1 < \frac{n^{\frac{3}{2}} M}{n^{2.1}} \qquad (n \geqq 4),$$

$$\int_{-1}^{1} n^{\frac{3}{2}} P_n\, dt_0 = 0 \qquad (n \geqq 1).$$

* But I cannot show that for every continuous function A_n remains limited.

† Exactly, $\phi = [(2n+1)/2n]\, P_n$, but this approaches the same limit as P_n. Similar remarks hold for the other proofs.

Proof of (f).—Consider for example G_5'. Using Theorem 3 for G_5' and $\phi = n^{2.1} P_n$, and Lemma 2 (b) and (a), we have

$$\left| \int_{-1}^{t_2} \left(\int_{-1}^{t_1} n^{2.1} P_n \, dt_0 \right) dt_1 \right| < \frac{n^{2.1} M}{n^{2.1}} = M \qquad (n \geqq 4),$$

$$< \frac{n^{2.1} M}{n^{2.5} (1 - t_2^2)^{\frac{1}{4}}} \quad \text{if } t_2^2 \neq 1 \quad (n \geqq 4),$$

$$= 0 \quad \text{if } t_2^2 = 1 \quad (n \geqq 2),$$

the last result being obtained by use of the proof of Lemma 2. Similarly,

$$\int_{-1}^{1} n^{2.1} P_n \, dt_0 = 0 \qquad (n \geqq 1).$$

COROLLARY 1: *If f is the integral of a function which has limited variation in the interval $(-1, 1)$, its Legendre development converges absolutely uniformly in the same interval.*

For, by (f), the absolute value series may be compared with the series whose general term is $1/n^{1.1}$.

COROLLARY 2: *If f has limited variation in $(-1, 1)$, the series whose general term* is $|A_n| P_n^2$ converges uniformly in any interval within $(-1, 1)$.*

For, by (2), in any such interval nP^2 remains uniformly limited, and therefore, by (d), the series in question may be compared with the series whose general term is $1/n^{1.1}$.

We may now apply to the developments of Legendre the same reasoning as was used in the case of Fourier's developments, and obtain the following results concerning their rates of convergence:†

COROLLARY 3: *Let R_n be the remainder of the formal development of f at the point x, and let $\psi(t)$ be the fraction $[f(t) - f(x)]/(t - x)$.*

(a) If ψ is absolutely L-integrable, R_n/n approaches zero.

(b) If ψ^2 is L-integrable, $R_n/\sqrt{n}$ approaches zero.

(c) If ψ has limited variation, $n^{0.1} R_n$ approaches zero.

(d) If ψ belongs to $G_5^{(i)}$, $n^{i+0.1} R_n$ approaches zero.

9. **Definite Integrals.**—Much of this section, in a somewhat different form, has been established by W. H. Young‡ independently of Lebesgue's theorems, but by means of reasoning closely analogous to that of Lebesgue. The present paper shows that the results are really only corollaries of Lebesgue's

* À fortiori this is true of the series whose general term is $A_n^2 P_n^2$.

† Cf. D. Jackson, *On the Degree of Convergence*, etc., these Transactions, vol. 13 (1912), pp. 305–318; and *On Approximation by Trigonometric Sums*, etc., ibid., pp. 491–515.

‡ Young, Proceedings of the London Mathematical Society, ser. 2, vol. 9 (1910), pp. 463–485.

fundamental theorem (§ 2), and possesses the advantage that the conditions stated here are thus shown to be necessary as well as sufficient. Young has nothing to do with the family F_6, and in Theorem 6 I consider several families where Young considers but one. His account includes, however, two topics which mine omits: non-absolutely convergent L-integrals, and integrals over infinite intervals.

Let us consider the evaluation of the definite integral

$$\int_k^l f(t)\, g(t)\, dt, \tag{1}$$

by means of the common method of expanding $g(t)$ in a series,

$$g(t) = u_1(t) + u_2(t) + \cdots \tag{2}$$

(the sum of the first n terms of which shall be called $s_n(t)$), multiplying it through by $f(t)$, and integrating termwise from k to l, thus:

$$\int_k^l fg\,dt = \int_k^l fu_1\, dt + \int_k^l fu_2\, dt + \cdots. \tag{3}$$

Concerning the validity of this process we have the following

THEOREM 5: *Let the various integrals of* (3) *exist as absolutely convergent L-integrals. A necessary and sufficient condition that* (3) *shall be valid for all functions f belonging to one of the families F of Theorems* 1 *and* 2, *is that* $g - s_n$ *shall satisfy the conditions imposed on* ϕ *in that part of those theorems that applies to the F in question.*

In the cases of* F_1, F_2, *and* F_3, *the condition stated remains sufficient if* (2°) *of Theorem* 1 *be replaced by the condition that* (2) *be valid in* (k, l), *except perhaps at a null set.*

For it is necessary and sufficient that

$$\int_k^l (g - s_n) f\, dt$$

approach zero, and therefore Theorem 1 (with corollary) and Theorem 2 are applicable.

Continuing the same problem and notation, let us now suppose that it is not convenient to express g or its 1st, 2d, . . . , or $(i-1)$th integral $(i \gtreqless 1)$ by means of a series, but that its ith integral may be so expressed, thus:

$$J(t_i) = \int_k^{t_i} \cdots \int_k^{t_i} g\, dt_0 = U_1(t_i) + U_2(t_i) + \cdots. \tag{1}$$

Writing

$$S_n = U_1 + \cdots + U_n, \qquad R_n = J - S_n, \tag{2}$$

* Stated in another paper (see second footnote), pp. 63, 64.

and denoting differentiation by primes, we have, except at most at a null set,

$$J^{(i)} = g = s_n + r_n. \tag{3}$$

We now give the conditions that

$$\int_k^l fg\,dt = \lim_{n=\infty} \int_k^l s_n\,dt; \tag{4}$$

that is, that (1) may be differentiated termwise i times, then multiplied through by f and integrated termwise from k to l, giving the integral of fg.

THEOREM 6: *Let* ($a°$) *the U's have i derivatives, and let all the integrals involved exist as absolutely convergent L-integrals. Let* ($b°$) $\lim_n R_n^{(p)}(k) = \lim_n R_n^{(p)}(l) = 0$, $p = 0, 1, \cdots, i-1$. *Then a necessary and sufficient condition that* (4) *be valid, for all functions belonging to the family* $G_j^{(i)}$, *is that* R_n *satisfy the conditions on* ϕ *in Theorem* 1, *or* 2, *for* $F_j^{(i)}$.

In the cases of $G_1^{(i)}$, $G_2^{(i)}$, *and* $G_3^{(i)}$, *the conditions remain sufficient if it be required that* (1) *be valid in* (k, l), *except perhaps at a null set, instead of that* R_n *satisfy* (2°) *of Theorem* 1.

Proof.—From ($a°$) and successive applications of (d), § 4, it follows that

$$\begin{aligned}\int_k^{t_{i-p}} \cdots \int_k^{t_1} r_n\,dt_0 &= R_n^{(p)}(t_{i-p}) + [P_0 R_n^{(p)}(k) + \cdots + P_{i-p-1} R_n^{i-1}(k)] \\ &= R_n^{(p)} + \eta \qquad (p = 0, \cdots, i-1),\end{aligned} \tag{5}$$

where P_s is a polynomial of degree s in t_{i-p}. By ($b°$), η approaches zero uniformly with respect to t_{i-p}. It is necessary and sufficient that

$$\lim_{n=\infty} \int_k^l f r_n\,dt = 0, \tag{6}$$

and for this the conditions of Theorem 3 relative to $G_j^{(i)}$ are necessary and sufficient. To see that the theorem is true it is now only necessary to write out the conditions involved in that theorem and in our present hypothesis in full. This will be done for the case of $G_1^{(i)}$ only. Theorem 3 says that the first of the following integrals shall remain uniformly limited if n is large enough (except perhaps at a null set), and that the others shall approach zero.

$$\int_k^{t_i} \cdots \int_k^{t_1} r_n\,dt_0, \tag{7}$$

$$\int_k^{t_{i+1}} \cdots \int_k^{t_1} r_n\,dt_0, \tag{8}$$

$$\int_k^l \int_k^{t_{i-1}} \cdots \int_k^{t_1} r_n\,dt_0,\ \int_k \int_k^{t_{i-2}} \cdots \int_k^{t_1} r_n\,dt_0,\ \cdots,\ \int_k^l r_n\,dt_0. \tag{9}$$

Sufficiency.—By ($b°$) $\lim_{n=\infty} R_n^{(p)}(l) = 0$, $p = 0, \cdots, i-1$, and this with (5) shows that (9) approach zero. By hypothesis and (1°) of Theorem 1, $|R_n(t_i)| < M$, $n > n_M$, except perhaps at a null set, and this with (5) shows that the same is true of (7), and again, by (2°) of Theorem 1,

$$\lim_{n=\infty} \int_k^{t_{i+1}} R_n \, dt_i = 0,$$

and therefore, since η approaches zero uniformly, (8) approaches zero.

Necessity.—If either (1°) or (2°) of Theorem 1 is not satisfied by R_n, it follows from (5) that either (7) or (8), both necessary conditions, is not satisfied.

The corollary of Theorem 1 applies to the second part of the theorem.

10. **Termwise Integration.**—We might now consider the following similar problem. Let g be differentiable i times, and let its ith derivative be represented by a series. Under what conditions may this series be integrated termwise i times, then multiplied through by f and again integrated termwise, giving the integral of fg? But this method of evaluating a definite integral would not ordinarily be of advantage, because it is integration and not differentiation that usually leads to simpler functions, though it does frequently happen that it is no *harder* to evaluate an integral in this way than in one of the others, for what is lost in making the expansion more difficult is gained from the extra generality of the function f by which its integral may be multiplied. However, I wish to present the matter in another form. As already noted, a necessary and sufficient condition that termwise integration of a convergent sequence $\{\phi(n)\}$ be permissible is (1°) of Theorem 3. Let us suppose that ϕ satisfies not only this condition but also the severer condition (1°) of Theorem 1. Is anything more than termwise integration permissible? An answer to this question is found in Theorem 7, for which I introduce the following definition and lemmas.

DEFINITION: Let us denote by $E_j^{(i)}(x)$ the family of functions f such that, for a certain preassigned x in (k, l), $f(t)(t-x)^i$ belongs to F_j, i being a positive integer or zero.*

LEMMA 1: *In order that the integral of Theorem* 1 *shall exist and approach zero for all functions f belonging to the family* $E_j^{(i)}$, *it is necessary and sufficient that the conditions of Theorem* 1 *for* F_j *be satisfied when* ϕ *is replaced by* $\phi/(t-x)^i$.

This is a simple corollary of Theorem 1.

LEMMA 2: *If* $g(t)$ *is continuous in* (k, l), *the ith iterated integral* $J(t_i)$ *may be expressed thus*:

$$(1) \qquad J(t_i) = \int_x^{t_i} \cdots \int_x^{t_1} g \, dt_0 = \frac{(t_i - x)^i}{i!} g(\tau),$$

where τ *is a point of the interval* (x, t_i).

* E. g., $t^{-\frac{1}{2}}$ belongs to $E_1'(0)$ in $(0, 1)$. As far as Lemma 1 is concerned i might be any real number.

For, by successive applications of (a), § 4, we may learn that, for each p $(p = 0, 1, \cdots, i-1)$ and at each point of (k, l),

$$\frac{d^p J}{dt^p} = \int_x^{t_{i-p}} \cdots \int_x^{t_1} g\, dt_0, \qquad \frac{d^i J}{dt^i} = g. \tag{2}$$

By a well-known theorem we may therefore write

$$J(t_i) = J(x) + \frac{dJ}{dt}\Big|_x \frac{t_i - x}{1!} + \cdots + \frac{(t_i - x)^i}{i!} g(\tau),$$

and, evidently, by (1) and (2), all the terms on the right vanish except the last.

THEOREM 7: *If, in the interval (k, l), (1°) of Theorem 1 is satisfied, and $\{\phi(n)\}$ is a convergent sequence having continuous terms and a continuous limit, it is permissible (a) to integrate $\{\phi(n)\}$ termwise i times from x to t, and (b) to multiply both sides of the resulting equation by any function f belonging to $E_1^{(i)}(x)$, and again to integrate termwise from a to b, a, b, and x being any points of the interval, thus:*

$$(a) \qquad \lim_{n=\infty} \int_x^{t_i} \cdots \int_x^{t_1} \phi(n)\, dt_0 = \int_x^{t_i} \cdots \int_x^{t_1} \lim_{n=\infty} \phi(n)\, dt_0,$$

$$(b) \quad \lim_{n=\infty} \int_a^b f(t_i)\, dt_i \int_x^{t_i} \cdots \int_x^{t_1} \phi(n)\, dt_0 = \int_a^b f(t_i)\, dt_i \int_x^{t_i} \cdots \int_x^{t_1} \lim_{n=\infty} \phi(n)\, dt_0.$$

Proof.—Without loss of generality we may suppose that $\lim \phi(n) = 0$, for otherwise the limit of the sequence $\{\phi(n) - \lim \phi(n)\}$ is zero, and this latter sequence possesses the properties ascribed to $\{\phi\}$ above.

Case 1: $(i = 0)$. We have to show merely that, for f in $(E_1^v = F_1)$,

$$(b) \qquad \lim_{n=\infty} \int_a^b f\phi(n)\, dt = 0;$$

and this has been established already in the corollary to Theorem 1.

Case 2: $(i > 0)$. Let us suppose that (b) is true when i is replaced by $r - 1$; if we choose $f = 1$, the result is (a) for $i = r$. Hence (a) may be obtained by induction from (b) for Case 1. To establish (b) for Case 2 it is, by Lemmas 1 and 2, necessary and sufficient that

$$\lim_{n=\infty} \int_a^b f(t_i)\, \phi(n, \tau)\, \frac{(t_i - x)^i}{i!}\, dt_i = 0.$$

From the corollary to Theorem 1 and Lemma 1, it is clear that for this it is sufficient that $\phi(n, \tau)$ be uniformly limited in (a, b), and that for each τ its limit be zero. Both these conditions are satisfied by hypothesis.

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CONGRUENCES AND COMPLEXES OF CIRCLES*

BY

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The differential geometry of circle systems has received a large amount of attention of late. The subject has been approached from two quite distinct sides. On the one hand we have Koenigs, Cosserat, Moore, Bompiani and others who fix their attention on what we may call the descriptive differential properties of circles. The totality of circles in three-dimensional space may be represented by an S_6^5, lying in an S_9, and this variety may be studied by the now familiar methods of projective differential geometry. The theorems so reached are invariant under the twenty-four parameter group of sphere transformations. The other class of writers, wherein we may include Bianchi, Tzitzeica, Guichard and Eisenhart, have confined themselves largely to congruences (two-parameter systems) of circles, frequently to normal congruences. The methods employed have been the general ones of differential geometry and the center of interest has been rather more in the axes of the circles than the circles themselves.

It has seemed for some time to the present writer that the last word on these subjects had not by any means been written, and that by a different method of approach not only might we obtain simpler proofs of known theorems but discover a number of new theorems as well. The most interesting properties of circles are those which are invariant under inversion, or under the ten-parameter group of conformal transformations of space; the best approach to this group is through the use of pentaspherical coördinates. It is true that some of the writers mentioned above have made use of these coördinates, and still more has been done by Darboux in his *Théorie des Surfaces*, yet the possibilities of these coördinates have been by no means exhausted, and in the present paper they are more systematically applied to problems in differential circle geometry than has been the case in the past.

A circle may be regarded in two different aspects, either as a locus of points, or an envelop of spheres. The two points of view are, of course, closely related, but the change of emphasis leads naturally to rather different sets of theorems. The first section of the present article is devoted to preliminary formulæ for points and spheres in pentaspherical coördinates, and certain

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fundamental theorems concerning cospherical and orthogonal circles. In section 2 a parametric representation is given of the points of a circle, and it is shown how this leads to direct and elegant proofs of the usual theorems about circle congruences, in so far as they are invariant under the conformal group. In the third section the circle is considered in its other aspect, and the classic methods of the Kummer line-geometry are applied to circle congruences, thus giving a goodly number of new theorems. The fourth section is devoted to three-parameter families of circles.

§ 1. Preliminary Formulæ.

In the present work we shall confine ourselves to points in the finite region of space, locating each by a system of homogeneous pentaspherical coördinates, whose relation to the homogeneous rectangular Cartesian system is exhibited by the following equations:

$$
\begin{aligned}
&\rho x_0 = i\,(x^2 + y^2 + z^2 + t^2), && \sigma x = x_2,\\
&\rho x_1 = x^2 + y^2 + z^2 - t^2, && \sigma y = x_3,\\
&\rho x_2 = 2xt, && \sigma z = x_4,\\
&\rho x_3 = 2yt, && \sigma t = -\,(ix_0 + x_1),\\
&\rho x_4 = 2zt, &&
\end{aligned} \tag{1}
$$

$$
\sum_{i=0}^{i=4} x_i^2 \equiv (xx) = 0. \tag{2}
$$

A sphere will be given by an equation of the form

$$
\sum_{i=0}^{i=4} y_i x_i \equiv (yx) = 0. \tag{3}
$$

The sphere will be null if (y) satisfy the identity (2) in which case y_j will be the coördinates of the center in our present system. The angle of two not null spheres (y) and (z) will be

$$
\theta = \cos^{-1} \frac{(yz)}{\sqrt{(yy)}\,\sqrt{(zz)}}. \tag{4}
$$

It is convenient when a sphere is not null to assume that its coördinates (coefficients) satisfy the identity

$$
(yy) = 1. \tag{5}
$$

If (x) and (x') be two points, then every sphere (y), where

$$
y_i = \lambda x_i + \mu x_i',
$$

will be a sphere of inversion to interchange the two. We see, in fact, that every sphere orthogonal to (x) and (x') is orthogonal to (y) also. By our identity,

$$(xx) = (x'x') = 0.$$

If then

$$(xx') = 0$$

the sphere (y) will be null. The circle common to the null spheres whose centers are (x) and (x') will contain none but null spheres, i. e., (x) and (x') lie on an isotropic line. Suppose that we have two infinitely near points (x) and $(x + dx)$,

$$(xdx) = -\tfrac{1}{2}(dxdx).$$

Since this is a differential expression of the second order, we may take (dx) as the coördinates of the sphere through (x) orthogonal to the direction of advance to $(x) + (dx)$. The condition that two directions of advance should be mutually perpendicular is thus

$$(dx\delta x) = 0. \tag{6}$$

If a direction of advance be isotropic we have

$$(dxdx) = 0. \tag{7}$$

Through each circle of finite radius will pass two null spheres whose centers are called the *foci* of the circle. The necessary and sufficient condition that a circle should be null is that the foci should fall together. If, thus, the circle be given by the spheres (y) and (z) it will be null if we have equal roots in the quadratic

$$\lambda^2(yy) + 2\lambda\mu(yz) + \mu^2(zz) = 0,$$

that is,

$$(yy)(zz) - (yz)^2 = 0.$$

The following well-known theorems are given for reference:

A. A necessary and sufficient condition that two circles of finite radius should be cospherical is that their foci should be concyclic.

B. A necessary and sufficient condition that two circles of finite radius should touch one another is that their foci should lie on two intersecting isotropic lines.

C. If two circles be cospherical they will be cospherical and orthogonal to an infinite number of circles; if they be not cospherical and their foci do not lie in pairs on two isotropics, there will be two circles cospherical and orthogonal to both. The last two circles are in bi-involution, i. e., every sphere through the one is orthogonal to every sphere through the other.

There remains the case where the foci of the two circles lie in pairs on two skew isotropics. These two will be generators of one system of a sphere.

There is a one-parameter group of conformal transformations leaving invariant these two generators as well as all generators of the other system. For instance, if the fixed sphere were

$$x_0 = 0$$

we might consider transformations of the quaternion type

$$(x_1' + ix_2' + jx_3' + kx_4') = (a + bi + cj + dk)(x_1 + ix_2 + jx_3 + kx_4),$$
$$i^2 = j^2 = k^2 = ijk = -1.$$

It appears thus that circles cospherical and orthogonal to our given circles will be continuously transformed by this group, so that there are an infinite number of such circles. Our original circles shall be said in this case to be paratactic.*

D. A necessary and sufficient condition that two non-cospherical circles should be cospherical and orthogonal to more than two circles is that they should be paratactic.

§ 2. The Parametric Method. The Classical Theorems of Ribaucour and Darboux.

Let a circle be determined by a sphere (β) to which it is orthogonal, and the two points of intersection therewith (α) and (γ). The coördinates of every point thereon may be exhibited in the parametric form

$$x_i = \rho^2 \alpha_i + \rho\beta_i + \gamma_i,$$

$$(\alpha\alpha) = (\alpha\beta) = (\beta\gamma) = (\gamma\gamma) = 0, \tag{8}$$

$$(\beta\beta) = -2(\alpha\gamma) = 1.$$

This form is especially suitable to the study of congruences of circles. We therefore assume that $(\alpha), (\beta), (\gamma)$ are functions of two independent variables u and v, the various ratios being naturally supposed not to be all constant or functions of a single variable. We have the additional equations

$$\left(\alpha\frac{\partial\alpha}{\partial u}\right) = \left(\alpha\frac{\partial\alpha}{\partial v}\right) = \left(\beta\frac{\partial\beta}{\partial u}\right) = \left(\beta\frac{\partial\beta}{\partial v}\right) = \left(\gamma\frac{\partial\gamma}{\partial u}\right) = \left(\gamma\frac{\partial\gamma}{\partial v}\right) = 0,$$

$$\left(\alpha\frac{\partial\beta}{\partial u}\right) + \left(\beta\frac{\partial\alpha}{\partial u}\right) = \left(\alpha\frac{\partial\beta}{\partial v}\right) + \left(\beta\frac{\partial\alpha}{\partial v}\right) = \left(\beta\frac{\partial\gamma}{\partial u}\right) + \left(\gamma\frac{\partial\beta}{\partial u}\right) = \tag{9}$$

$$\left(\beta\frac{\partial\gamma}{\partial v}\right) + \left(\gamma\frac{\partial\beta}{\partial v}\right) = \left(\alpha\frac{\partial\gamma}{\partial u}\right) + \left(\gamma\frac{\partial\alpha}{\partial u}\right) = \left(\alpha\frac{\partial\gamma}{\partial v}\right) + \left(\gamma\frac{\partial\alpha}{\partial v}\right) = 0.$$

Our groups of equations (8) and (9) lead to easy proofs of all the classical

* The concept of paratactic circles seems first to have been introduced in an article by the Author, *A Study of the Circle Cross*, These Transactions, vol. 14 (1913), pp. 149–174. Proofs of our theorems *A–D* will there be found.

theorems about circle congruences, in so far, that is, as these theorems are invariant for inversion. We state a number of these theorems which are particularly important. They will be found for the most part in the second volume of Darboux's *Theorie generale des surfaces*, but mostly trace their origin to Ribaucour.*

THEOREM 1. *The circles of a congruence are, in general, tangent to four surfaces.*

In practice it is more convenient to say that a circle in general position intersects four adjacent circles. When this intersection arises from the fact that it is cospherical with two adjacent circles, i. e., strictly speaking, when each circle is a generator of two annular surfaces of the congruence, the latter is said to be *focal*. The corresponding generating spheres through the circle are called the *focal spheres*. The points where the circle touches the four surfaces are the *focal points*, and must not be confused with the *foci* mentioned above. If the circles of a congruence be orthogonal to an infinite number of surfaces, the congruence is said to be *normal*. In counting the number of orthogonal trajectories attached to a given congruence, we count the number of points on each circle where it meets some surface orthogonally. Thus, the circles orthogonal to a given sphere are said to have (at least) *two* orthogonal trajectories.

THEOREM 2. *If the circles of a congruence be orthogonal to more than two surfaces, the congruence is normal.*

THEOREM 3. *Any four orthogonal trajectories of the circles of a normal congruence will meet those circles in sets of points having a fixed cross ratio.*

THEOREM 4. *The foci of the circles of a focal congruence generate the two nappes of the envelope of a congruence of spheres.*

THEOREM 5. *If the foci of the circles of a congruence be the pairs of points of contact of the spheres of a non-parabolic congruence with their envelope, the congruence of circles is focal.*

THEOREM 6. *A necessary and sufficient condition that a focal congruence should be normal is that the focal spheres through a circle in general position should be mutually orthogonal, or that it should consist in circles through two points.*

THEOREM 7. *Every normal congruence is focal.*

THEOREM 8. *In a normal congruence the lines of curvature correspond to one another in all orthogonal trajectories, and give the annular surfaces of the congruence.*

THEOREM 9. *If a congruence of spheres have an envelope of two nappes whereon the lines of curvature correspond to one another, the circles orthogonal to the various spheres at their points of contact will generate a normal congruence.*

* See his two notes *Sur les systemès cyclicques* and *Sur les faisceaux de cercles*, Paris, Comptes Rendus, vol. 76 (1873); also an earlier note *Sur la déformation des surfaces*, ibid., vol. 70 (1870).

THEOREM 10. *If the circles of a congruence have two orthogonal trajectories the congruence will be focal when, and only when it is normal.*

THEOREM 11. *If the circles of a congruence be normal to two surfaces which do not consist in a sphere counted twice, and can be assembled into a one-parameter family of annular surfaces, the congruence is normal.*

As examples of the method of applying our equations (8), (9) to the proof of these theorems, we select theorems 2, 3, 6, and 8.

2. We wish to make t in (8) such a function of u and v that if dx_i be the differential along this surface, and δx_i the differential along the circle, then

$$(dx\,\delta x) = 0.$$

This involves

$$-\frac{\partial t}{\partial u} = \left(\alpha\frac{\partial\beta}{\partial u}\right)t^2 + 2\left(\alpha\frac{\partial\gamma}{\partial u}\right)t + \left(\beta\frac{\partial\gamma}{\partial u}\right),$$

$$-\frac{\partial t}{\partial v} = \left(\alpha\frac{\partial\beta}{\partial v}\right)t^2 + 2\left(\alpha\frac{\partial\gamma}{\partial v}\right)t + \left(\beta\frac{\partial\gamma}{\partial v}\right).$$

The condition for compatibility becomes, in the light of (9),

$$\begin{aligned}(10)\quad &\left\{\left(\frac{\partial\alpha}{\partial v}\frac{\partial\beta}{\partial u}\right) - \left(\frac{\partial\alpha}{\partial u}\frac{\partial\beta}{\partial v}\right) - 2\left[\left(\alpha\frac{\partial\beta}{\partial u}\right)\left(\alpha\frac{\partial\gamma}{\partial v}\right) - \left(\alpha\frac{\partial\beta}{\partial v}\right)\left(\alpha\frac{\partial\gamma}{\partial u}\right)\right]\right\}t^2 \\ &+ 2\left\{\left(\alpha\frac{\partial\beta}{\partial v}\right)\left(\beta\frac{\partial\gamma}{\partial u}\right) - \left(\alpha\frac{\partial\beta}{\partial u}\right)\left(\beta\frac{\partial\gamma}{\partial v}\right) + \left(\frac{\partial\alpha}{\partial v}\frac{\partial\gamma}{\partial u}\right) - \left(\frac{\partial\alpha}{\partial u}\frac{\partial\gamma}{\partial v}\right)\right\}t \\ &+ \left\{\left(\frac{\partial\beta}{\partial v}\frac{\partial\gamma}{\partial u}\right) - \left(\frac{\partial\beta}{\partial u}\frac{\partial\gamma}{\partial v}\right) + 2\left[\left(\alpha\frac{\partial\gamma}{\partial v}\right)\left(\beta\frac{\partial\gamma}{\partial u}\right) - \left(\alpha\frac{\partial\gamma}{\partial u}\right)\left(\beta\frac{\partial\gamma}{\partial v}\right)\right]\right\} = 0.\end{aligned}$$

Since this equation is quadratic, if it have more than two solutions it is an identity.

3. Let (α) and (γ) be so chosen as to trace two orthogonal trajectories

$$(11)\qquad (\alpha d\beta) = (\beta d\alpha) = (\gamma d\beta) = (\beta d\gamma) = 0.$$

For another orthogonal trajectory we must have

$$-\frac{\partial t}{\partial u} = 2\left(\alpha\frac{\partial\gamma}{\partial u}\right)t, \qquad -\frac{\partial t}{\partial v} = 2\left(\alpha\frac{\partial\gamma}{\partial v}\right)t.$$

The condition of compatibility is

$$(12)\qquad \left(\frac{\partial\alpha}{\partial u}\frac{\partial\gamma}{\partial v}\right) = \left(\frac{\partial\alpha}{\partial v}\frac{\partial\gamma}{\partial u}\right).$$

This equation will be of great importance later. If t be one solution of our system of differential equations above, rt is another, if r be constant. But

this constant is the cross ratio of the four points corresponding to the parameter values $\infty, 0, t, rt$.

6. Our congruence being focal, we may take u and v as focal parameters. We take for (β) the sphere whose points of contact with its envelop will, by theorem 4, be the foci of our circles; these foci shall be (s) and (t)

$$(ss) = (s\alpha) = (s\beta) = (s\gamma) = (sd\beta) = 0,$$

$$(tt) = (t\alpha) = (t\beta) = (t\gamma) = (td\beta) = 0.$$

From the first equations we get

$$A\alpha_i + B\beta_i + C\gamma_i + D\frac{\partial \beta_i}{\partial u} + Es_i = 0,$$

$$A'\alpha_i + B'\beta_i + C'\gamma_i + D'\frac{\partial \beta_i}{\partial v} + E's_i = 0.$$

Multiplying through by t_i and summing, then by β_i and summing, we have

$$E = B = 0, \qquad D \neq 0,$$

$$\frac{\partial \beta_i}{\partial u} = a\alpha_i + c\gamma_i, \qquad \frac{\partial \beta_i}{\partial v} = a'\alpha_i + c'\gamma_i.$$

Since all points of the circle $(\alpha) + (\partial\alpha/\partial u)\,du$, $(\gamma) + (\partial\gamma/\partial u)\,du$, $(\beta) + (\partial\beta/\partial u)\,du$ lie on a sphere through our circle, a focal sphere,

$$\frac{\partial \gamma_i}{\partial u} = l\alpha_i + m\beta_i + n\gamma_i + r\frac{\partial \alpha_i}{\partial u},$$

$$\frac{\partial \gamma_i}{\partial v} = \lambda\alpha_i + \mu\beta_i + \nu\gamma_i + \rho\frac{\partial \alpha_i}{\partial v}.$$

If the congruence be normal, (10) must be satisfied identically. On substituting, the coefficient of t^2 and the constant vanish. The condition is thus

$$\left[cc' - 4\left(\frac{\partial \alpha}{\partial u}\frac{\partial \alpha}{\partial v}\right)\right](r - \rho) = 0.$$

If $r = \rho$, every solution of

$$(x\alpha) = (x\beta) = (x\gamma) = (xd\alpha) = 0$$

is also a solution of

$$(xd\beta) = (xd\gamma) = 0.$$

Each two adjacent circles are co-spherical, and all of our circles pass through two points or lie on one sphere. The latter would not however yield a normal congruence.

If $r \neq \rho$ we may take as focal spheres

$$\rho y_i = \left| \alpha_j \beta_k \gamma_l \frac{\partial \alpha_m}{\partial u} \right|, \qquad \sigma z_i = \left| \alpha_j \beta_k \gamma_l \frac{\partial \alpha_m}{\partial v} \right|.$$

The condition that these should be mutually orthogonal is

$$cc' - 4\left(\frac{\partial \alpha}{\partial u}\frac{\partial \alpha}{\partial v}\right) = 0.$$

8. We have a normal congruence, so that we have the use of equations (11). We may in two ways (usually distinct) solve the following equations for s_i, du, dv

$$(s\alpha) = (s\beta) = (s\gamma) = (sd\alpha) = (sd\gamma).$$

By a change of parameters we may write

$$\frac{\partial \gamma_i}{\partial u} = l\alpha_i + b\beta_i + m\gamma_i + n\frac{\partial \alpha_i}{\partial u},$$

$$\frac{\partial \gamma_i}{\partial v} = \lambda\alpha_i + b'\beta_i + \mu\gamma_i + \rho\frac{\partial \alpha_i}{\partial u}.$$

Multiplying through by β_i and summing, we get

$$b = b' = 0.$$

The condition for a normal congruence (12) gives

$$n\left(\frac{\partial \alpha}{\partial u}\frac{\partial \alpha}{\partial v}\right) = \rho\left(\frac{\partial \alpha}{\partial u}\frac{\partial \alpha}{\partial v}\right).$$

If $n = \rho$, then

$$d\gamma_i = P\alpha_i + Q\gamma_i + Rd\alpha_i.$$

The points (α), (γ), $(\alpha) + (d\alpha)$, $(\gamma) + (d\gamma)$ are always concyclic. Hence the circles with these as foci pass through two points or lie on a sphere, since any two adjacent ones are cospherical. The first case is inadmissible as (α) and (γ) would trace a circle. In the other case they trace surfaces mutually inverse in this sphere and the lines of curvature correspond.

If $n \neq \rho$, then

$$\left(\frac{\partial \alpha}{\partial u}\frac{\partial \alpha}{\partial v}\right) = -\left(\alpha\frac{\partial^2 \alpha}{\partial u\,\partial v}\right) = 0,$$

$$\left(\frac{\partial \gamma}{\partial u}\frac{\partial \gamma}{\partial v}\right) = -\left(\gamma\frac{\partial^2 \gamma}{\partial u\,\partial v}\right) = 0.$$

The condition of compatibility of the two partial differential equations for (γ) is

$$(R-P)\frac{\partial^2\alpha_i}{\partial u\,\partial v}+\beta\frac{\partial\alpha_i}{\partial u}+C\frac{\partial\alpha_i}{\partial v}+D\alpha_i+E\gamma_i=0.$$

Multiplying through by α_i and summing, we find $E=0$ and

$$\frac{\partial^2\alpha_i}{\partial u\,\partial v}=a\alpha_i+b\frac{\partial\alpha_i}{\partial u}+c\frac{\partial\alpha_i}{\partial v},$$

$$\frac{\partial^2\gamma_i}{\partial u\,\partial v}=a'\alpha_i+b'\frac{\partial\alpha_i}{\partial u}+c'\frac{\partial\alpha_i}{\partial v}.$$

These equations are characteristic for the lines of curvature, for they show that the Gauss coefficient D' must vanish for both surfaces, while we saw above

$$\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right)=\left(\frac{\partial\gamma}{\partial u}\frac{\partial\gamma}{\partial v}\right)=0,$$

showing that the u and v curves are mutually orthogonal.*

§ 3. The Kummer Method.

Since the totality of spheres can be put into one-to-one correspondence with that of all points of a four-dimensional space (if planes be counted as spheres) so there is a one-to-one correspondence between the circles of our space and the lines of a four-dimensional space. Moreover our formula (4) for the angle of two spheres is identical with the non-euclidean distance formula, so that the angles of spheres will correspond to the non-euclidean distances of points. We are thus led to the idea of approaching our circle systems by the classical methods of Kummer for line geometry, with the modifications necessary for a four-dimensional universe, and non-euclidean measurement.† We shall fix a circle by two mutually orthogonal spheres (y) and (z) whose coördinates are supposed to be functions of u and v

$$y_i=y_i(u,v),\qquad z_i=z_i(u,v),\qquad (yy)=(zz)=1,\qquad (yz)=0.$$

We have the three fundamental equations

$$(dy\,dy)-(zdy)^2=Edu^2+2Fdu\,dv+Gdv^2,$$

$$(dz\,dz)-(ydz)^2=E'du^2+2F'du\,dv+G'dv^2, \tag{13}$$

$$(dy\,dz)=e\,du^2+(f+f')\,du\,dv+ydv^2.$$

More specifically

* Conf. Darboux, *Surfaces*, vol. I, p. 221.

† The Kummer formulæ for non-euclidean space were first worked out by Fibbi, *I sistemi doppiamente infiniti di raggi negli spazii di curvatura costante*, Annali della R. Scuola Normale Superiore, Pisa, 1891. See also the Author's *Elements of Non-Euclidean Geometry*, Oxford, 1909, Chap. 16.

$$\left(\frac{\partial y}{\partial u}\frac{\partial y}{\partial u}\right)-\left(z\frac{\partial y}{\partial u}\right)^2=E,\quad \left(\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\right)-\left(z\frac{\partial y}{\partial u}\right)\left(z\frac{\partial y}{\partial v}\right)=F,$$

$$\left(\frac{\partial y}{\partial v}\frac{\partial y}{\partial v}\right)-\left(z\frac{\partial y}{\partial v}\right)^2=G,$$

$$\left(\frac{\partial z}{\partial u}\frac{\partial z}{\partial u}\right)-\left(y\frac{\partial z}{\partial u}\right)^2=E',\quad \left(\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right)-\left(y\frac{\partial z}{\partial u}\right)\left(y\frac{\partial z}{\partial v}\right)=F', \tag{14}$$

$$\left(\frac{\partial z}{\partial v}\frac{\partial z}{\partial v}\right)-\left(y\frac{\partial z}{\partial v}\right)^2=G',$$

$$\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial u}\right)=e,\quad \left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)=f,\quad \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\right)=f',\quad \left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial v}\right)=g.$$

These various coefficients are connected by the following syzygy:

$$\begin{vmatrix} y_0 & \cdots & y_4 & 0 \\ z_0 & \cdots & z_4 & 0 \\ \frac{\partial y_0}{\partial u} & \cdots & \frac{\partial y_4}{\partial u} & 0 \\ \frac{\partial y_0}{\partial v} & \cdots & \frac{\partial y_4}{\partial v} & 0 \\ \frac{\partial z_0}{\partial u} & \cdots & \frac{\partial z_4}{\partial u} & 0 \\ \frac{\partial z_0}{\partial v} & \cdots & \frac{\partial z_4}{\partial v} & 0 \end{vmatrix}^2 = \begin{vmatrix} E & F & e & f' \\ F & G & f & g \\ e & f & E' & F' \\ f' & g & F' & G' \end{vmatrix} = 0. \tag{15}$$

We saw in theorem D that two circles in general position will be cospherical and orthogonal with two other circles, the latter being in bi-involution, i. e., all spheres through the one are orthogonal to all through the other. We start with two adjacent circles of our congruence; the circles orthogonal to them will be determined by two mutually orthogonal spheres through the first, each orthogonal to one of two mutually orthogonal spheres through the second. Let the spheres through the first circle be $\cos\varphi\,(y)+\sin\varphi\,(z)$, $-\sin\varphi\,(y)+\cos\varphi\,(z)$, while those through the second are $\lambda\,(y+dy)+u\,(z+dz)$, $-\mu\,(y+dy)+\lambda\,(z+dz)$; writing the conditions for criss-cross orthogonality we have two linear homogeneous equations in $\lambda:\mu$. Equating the discriminant to zero, and remembering that

$$(y\,dy)=-\tfrac{1}{2}(dy\,dy),\quad (z\,dz)=-\tfrac{1}{2}(dz\,dz),\quad (y\,dz)+(z\,dy)=-(dy\,dz),$$

we have

$$\begin{vmatrix} [1-\tfrac{1}{2}(dy\,dy)]\sin\varphi-(z\,dy)\cos\varphi & (y\,dz)\sin\varphi-[1-\tfrac{1}{2}(dz\,dz)]\cos\varphi \\ -[1-\tfrac{1}{2}(dz\,dz)]\sin\varphi-(y\,dz)\cos\varphi & (z\,dy)\sin\varphi+[1-\tfrac{1}{2}(dy\,dy)]\cos\varphi \end{vmatrix} = 0.$$

Expanding, and casting aside infinitesimals of higher order, we have

$$(16)\qquad \begin{aligned}&[e\,du^2+(f+f')\,du\,dv+g\,dv^2](\cos^2\varphi-\sin^2\varphi)+[(E-E')\,du^2\\&\quad+2(F-F')\,du\,dv+(G-G')\,dv^2]\sin\varphi\cos\varphi=0.\end{aligned}$$

In order to discuss this equation, we write the two

$$e\,du^2+(f+f')\,du\,dv+g\,dv^2=0,$$
$$(E-E')\,du^2+2(F-F')\,du\,dv+(G-G')\,dv^2=0.$$

They are equivalent to

$$(dy\,dz)=(dy\,dy)-(dz\,dz)=0.$$

Suppose then that the foci of our circle are (α), (γ). We may write

$$(17)\qquad \alpha_i=\frac{-\sqrt{-1}\,y_i+z_i}{2},\qquad \gamma_i=-\frac{\sqrt{-1}\,y_i+z_i}{2};$$
$$y_i=\sqrt{-1}\,(\alpha_i+\gamma_i),\qquad z_1=\alpha_i-\gamma_i.$$

Our pair of differential equations may thus be replaced by

$$(d\alpha\,d\alpha)=(d\gamma\,d\gamma)=0.$$

We have the following possibilities with regard to these two equations:

A. They have, in general, no common root. The isotropic curves do not correspond on the surfaces by the foci. We shall call these *non-conformal congruences.*

B. The equations have one common root, in general. Then on the two surfaces of foci (which must not be confused with the focal surfaces to which our circles are tangent) one system of isotropic curves will be in correspondence. These congruences shall be called *semi-conformal.*

C. The two equations are identical. The two surfaces of foci when they exist are conformally related, and the congruence is said to be *conformal.*

Let us begin with the non-conformal congruence. Here if $du:dv$ be given we usually get a unique value of $\tan 2\varphi$, that is two mutually orthogonal spheres, and so the two circles in bi-involution required. On the other hand if φ be given we have a quadratic in $du:dv$ so that on each sphere through a circle

in general position of a non-conformal congruence will lie two circles orthogonal to this circle, cospherical and orthogonal with two adjacent circles of the congruence. These circles will fall together if the equation in $du : dv$ have equal roots, i. e., if

$$\begin{aligned}(18)\quad &[eg - \tfrac{1}{2}(f+f')^2](\tan^2\varphi - 1)^2 \\ &+ [e(G-G') - (F-F')(f+f') + g(E-E')](\tan^2\varphi - 1)\tan\varphi \\ &+ [(E-E')(G-G') - (F-F')^2]\tan^2\varphi = 0.\end{aligned}$$

This equation is unaltered if we replace φ by $\pi/2 + \varphi$.

THEOREM 12. *Through each circle in general position of a non-conformal congruence will pass two pairs of mutually orthogonal spheres on each of which there is but one circle orthogonal to the given circle, cospherical and orthogonal to an adjacent circle of the congruence.*

These spheres shall be called the *limiting spheres.* They correspond to maximum and minimum values for $\tan\varphi$. We see, in fact, that if we equate to zero the partial derivatives of (16) with regard to du and dv, we get

$$\left[edu + \frac{f+f'}{2}dv\right](\tan^2\varphi - 1) - [(E-E')du + (F-F')dv]\tan\varphi = 0,$$

$$\left[\frac{f+f'}{2}du + gdv\right](\tan^2\varphi - 1) + [(F-F')du + (G-G')dv]\tan\varphi = 0.$$

Eliminating $du : dv$ we fall back on (18). In the case of a real congruence, the spheres containing real circles orthogonal and cospherical to the adjacent circle will lie in determinate angular openings between these limiting spheres. We next make the further assumption that our congruence is focal. The following equations are characteristic of a focal sphere

$$\cos\varphi\, y_i + \sin\varphi\, z_i \equiv \cos(\varphi + d\varphi)(y_i + dy_i) + \sin(\varphi + d\varphi)(z_i + dz_i),$$

$$dy_i \cos\varphi + dz_i \sin\varphi - (y_i \sin\varphi - z_i \cos\varphi)\, d\varphi = 0.$$

Multiplying through by y_i and summing, we have

$$d\varphi = (ydz) = -(zdy).$$

Substituting this value of $d\varphi$ in the last equation, we get

$$\left[\left(\frac{\partial y_i}{du}\cos\varphi + \frac{\partial z_i}{\partial u}\sin\varphi\right) - (y_i\sin\varphi - z_i\cos\varphi)\left(y\frac{\partial z}{\partial u}\right)\right]du$$

$$+ \left[\left(\frac{\partial y_i}{\partial v}\cos\varphi + \frac{\partial z_i}{\partial v}\sin\varphi\right) - (y_i\sin\varphi - z_i\cos\varphi)\left(y\frac{\partial z}{\partial v}\right)\right]dv = 0.$$

Multiplying through by $\partial z_i/\partial u$ and summing, then doing the same for $\partial z_i/\partial v$,

$$[edu + fdv]\cos\varphi + [E'du + F'dv]\sin\varphi = 0,$$
$$[f'du + gdv]\cos\varphi + [F'du + G'dv]\sin\varphi = 0.$$

Similarly

$$[edu + f'dv]\sin\varphi + [Edu + Fdv]\cos\varphi = 0,$$
$$[fdu + gdv]\sin\varphi + [Fdu + Gdv]\cos\varphi = 0.$$

Eliminating φ,

$$(E'f' - F'e)\,du^2 + [E'g - F'(f - f') - G'e]\,du\,dv + (F'g - G'f)\,dv^2 = 0, \tag{19}$$
$$(Ef - Fe)\,du^2 + [Eg - F(f' - f) - Ge]\,du\,dv + (Fg - Gf')\,dv^2 = 0.$$

Eliminating $du : dv$,

$$(E'G' - F'^2)\tan^2\varphi + [E'g - F'(f + f') + G'e]\tan\varphi + (eg - ff') = 0, \tag{20}$$
$$(eg - ff')\tan^2\varphi + [Eg - F(f + f') + Ge]\tan\varphi + (EG - F^2) = 0,$$

$$[(eg - ff') - (E'G' - F'^2)]\tan^2\varphi + [(E - E')g - (F - F') \times (f + f') + (G - G')e]\tan\varphi + [(EG - F^2) - (eg - ff')] = 0. \tag{21}$$

It is necessary, in order that a congruence should be focal, that the two equations (19) and the two equations (20) should be equivalent. It should be noted further that the middle coefficient is the same in (18) and (21). This coefficient will vanish when the corresponding pairs of spheres make equal angles with (y). Now if two spheres be coaxal with a third, and make equal angles therewith, they are mutually inverse therein and conversely, and we have

THEOREM 13. *In a focal congruence, the pairs of limiting spheres are mutually inverse in the same two spheres as are the two focal spheres.*

The necessary and sufficient conditions that a congruence should be focal are best reached in another form. Retaining (α) and (γ) for our foci there must, by (4) and (5), be one sphere through all the points (α), (γ), $(\alpha)+(d\alpha)$, $(\gamma) + (d\gamma)$. This gives the equations

$$\left|\frac{\partial\gamma}{\partial u}\ \alpha\ \gamma\ \frac{\partial\alpha}{\partial u}\ \frac{\partial\alpha}{\partial v}\right| = \left|\frac{\partial\gamma}{\partial v}\ \alpha\ \gamma\ \frac{\partial\alpha}{\partial u}\ \frac{\partial\alpha}{\partial v}\right| = 0,$$

which are equivalent to

$$\left|\frac{\partial z}{\partial u}\ y\ z\ \frac{\partial y}{\partial u}\ \frac{\partial y}{\partial v}\right| = \left|\frac{\partial z}{\partial v}\ y\ z\ \frac{\partial y}{\partial u}\ \frac{\partial y}{\partial v}\right| = 0.$$

Squaring we have

$$(22)\qquad \begin{vmatrix} E & F & e \\ F & G & f \\ e & f & E' \end{vmatrix} = \begin{vmatrix} E & F & f' \\ F & G & g \\ f' & g & G' \end{vmatrix} = 0.$$

The condition that two adjacent circles intersect is that their common orthogonal sphere should be null. This will give the equation

$$(Edu^2 + 2Fdu\,dv + Gdv^2)(E'\,du^2 + 2F'\,du\,dv + G'\,dv^2) - (edu^2 + (f+f')\,du\,dv + g'\,dv^2)^2 = 0.$$

The fact that this equation is of the fourth degree proves theorem 1. For a focal congruence, it must be the square of a quadratic expression. When we know that our congruence is focal, and take u and v as the focal parameters, we have the simple equations

$$\frac{\partial z_i}{\partial u} = ay_i + ac\,z_i + c\frac{\partial y_i}{\partial u}, \qquad \frac{\partial z_i}{\partial v} = a'\,y_i + a'\,c'\,z_i + c'\frac{\partial y_i}{\partial v},$$

$$(23)\qquad E' = ce, \qquad F' = cf' = c'f, \qquad G' = c'\,g,$$

$$E = \frac{1}{c}e, \qquad F = \frac{1}{c}f = \frac{1}{c'}f', \qquad G = \frac{1}{c'}g.$$

The focal spheres will be

$$c\,(y) - (z), \qquad c'\,(y) - (z).$$

The condition for a normal congruence is thus, by theorem 6,

$$cc' + 1 = 0.$$

This may be written in the invariant form

$$(24)\qquad (EG - F^2) = (E'\,G' - F'^2) = -(eg - ff').$$

These equations and (20) will give necessary and sufficient conditions for a normal congruence.

The normal congruence of circles seems superficially the most natural analog of the normal line congruence. From our present point of view, however, the natural analog is that circle congruence which corresponds to a normal congruence of lines in four-dimensional space. A little reflection shows that such a congruence has the following interpretation in circle geometry. Through each circle (perhaps only in a limited region) will pass one sphere whose points of contact with the two nappes of its envelop are mutually inverse in that circle. (The extension of inversion to the geometry on a sphere is immediate. Two points of a sphere are inverse in a circle thereof, if all

circles of the sphere through them are orthogonal to that circle.) Suppose that such a sphere have the coördinates

$$y_i' = \cos\varphi\, y_i + \sin\varphi\, z_i .$$

Then every sphere through the points of contact of (y') with its envelop cuts (y') in a circle orthogonal to the given circle, i. e., every such sphere is orthogonal to the sphere through the given circle orthogonal to (y'). In algebraic form, every sphere

$$\left[(y') + \left(\frac{\partial y'}{\partial u}\right)\right] du + \left(\frac{\partial y'}{\partial v}\right) dv$$

is orthogonal to the sphere

$$-\sin\varphi\,(y) + \cos\varphi\,(z).$$

This yields the equations

$$-\left(y\frac{\partial z}{\partial u}\right)\sin^2\varphi + \left(z\frac{\partial y}{\partial u}\right)\cos^2\varphi + \frac{\partial\varphi}{\partial u}$$

$$= -\left(y\frac{\partial z}{\partial v}\right)\sin^2\varphi + \left(z\frac{\partial y}{\partial v}\right)\cos^2\varphi + \frac{\partial\varphi}{\partial v} = 0, \tag{25}$$

$$\frac{\partial\varphi}{\partial u} = \left(y\frac{\partial z}{\partial u}\right), \qquad \frac{\partial\varphi}{\partial v} = \left(y\frac{\partial z}{\partial v}\right).$$

The condition of compatibility will be

$$f = f'. \tag{26}$$

The differential equations (25) are equally well solved by φ and $\varphi + k$ where k is any constant.

THEOREM 14. *If through each circle of a congruence it be possible to pass one sphere whose points of contact with the two nappes of the envelop are mutually inverse in that circle, then an infinite number of such spheres may be passed through each circle. The spheres generate a one-parameter family of congruences, the corresponding spheres of any two congruences will intersect at a constant angle.*

We shall say in this case that our congruence of circles is *pseudo-normal.* Suppose that we have a congruence which is of this sort, and also is focal. We find from (21)

$$(c - c')f = 0.$$

If

$$c = c',$$

$$dz_i = py_i + qz_i + rdy_i,$$

any two adjacent circles of the congruence will be cospherical, i. e., all the circles lie on a sphere, or pass through two points, two cases which we may rule out. Hence

$$f = f' = 0.$$

Solving (20) in view of (21) we find both roots are solutions of (18), hence

THEOREM 15. *The necessary and sufficient condition that a focal congruence should be pseudo-normal is that the focal spheres should coincide with a pair of limiting spheres.*

Let us see what relations subsist between the foci of the circles of a pseudo-normal congruence. We see from (17) that if we have

$$\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\right) = \left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right),$$

we also have

$$\left(\frac{\partial \alpha}{\partial u}\frac{\partial \gamma}{\partial v}\right) = \left(\frac{\partial \alpha}{\partial v}\frac{\partial \gamma}{\partial u}\right).$$

If then our congruence be focal as well as pseudo-normal, and (β) be the sphere whose points of contact with its envelop are (α) and (γ), by theorem 4 we see that the circle orthogonal to (β) at the points (α) and (γ) (and so in bi-involution with the given circle) will, by equation (12), generate a normal congruence. We next notice that if u and v be the focal parameters for the pseudo-normal congruence, our equations (23) lead to the equations used in proving theorem 8, so that they are focal parameters for the normal congruence also and give the lines of curvature of the (α) and (γ) surfaces.

THEOREM 16. *The necessary and sufficient condition that a focal congruence of circles should be pseudo-normal is that the lines of curvature should correspond in the surfaces traced by the foci of the circles.*

We may find still another necessary and sufficient condition for a pseudo-normal congruence. Revert to our equations (23); the focal spheres were

$$c\,(y) - (z), \qquad c'\,(y) - (z).$$

The spheres orthogonal to our circle through the pairs of focal points are

$$\left(\frac{\partial y}{\partial u}z\right)z_i - \frac{\partial y_i}{\partial u}, \qquad \left(\frac{\partial y}{\partial v}z\right)z_i - \frac{\partial y_i}{\partial v}.$$

These will be mutually orthogonal if

$$F = f = f' = 0.$$

When, however, two spheres are orthogonal to a circle and to one another, their pairs of intersections with the circle separate one another harmonically, and vice versa.

THEOREM 17. *The necessary and sufficient condition that a focal congruence of circles should be pseudo-normal is that the pairs of focal points should separate one another harmonically.*

We get at once from theorem 16

THEOREM 18. *If a normal congruence be given, the congruence of circles whose foci are the pairs of intersections of the circles of the normal congruence with any two orthogonal trajectories is focal and pseudo-normal.*

THEOREM 19. *If a congruence be focal and pseudo-normal, the foci are the pairs of intersections of the circles of a normal congruence with two orthogonal trajectories.*

When a normal and pseudo-normal congruence are related to one another in this way, we shall call them *associated.* The normal congruence has the parametric form

$$x_i = t^2 \alpha_i + t\beta_i + \gamma_i, \qquad \rho\beta_i = \left| y_j \; z_k \; \frac{\partial y_l}{\partial u} \frac{\partial y_m}{\partial v} \right|.$$

Here u and v are supposed to be the common focal parameters of both congruences. Let (y') be one of those spheres whose points of contact with the envelop are mutually inverse in the (y) (z) circle. We write

$$y_i' = \cos\varphi \, y_i + \sin\varphi \, z_i,$$

$$\frac{\partial y_i'}{\partial u} = \cos\varphi \frac{\partial y_i}{\partial u} + \sin\varphi \frac{\partial z_i}{\partial u} + \left(y \frac{\partial z}{\partial u} \right) (- y_i \sin\varphi + z_i \cos\varphi),$$

$$\frac{\partial y_i'}{\partial v} = \cos\varphi \frac{\partial y_i}{\partial v} + \sin\varphi \frac{\partial z_i}{\partial v} + \left(y \frac{\partial z}{\partial v} \right) (- y_i \sin\varphi + z_i \cos\varphi),$$

as we see with the aid of (25). We also see from (17) and the above value of (β) that all points of the circle of the normal congruence lie on the spheres $(\partial y'/\partial u)$, $(\partial y'/\partial v)$, so that, in particular, the points of contact of (β) with its envelop lie on this circle. The circle is orthogonal to (y'), since $(\partial y'/\partial u)$ and $(\partial y'/\partial v)$ are orthogonal thereto.

THEOREM 20. *If a focal and pseudo-normal congruence be given, the spheres whose pairs of points of contact with their envelops are mutually inverse in the circles of the congruence will generate the orthogonal trajectories of the circles of the associated normal congruence.*

Since our two congruences have the same focal parameters

THEOREM 21. *If a normal and pseudo-normal congruence be associated, the annular surfaces will correspond in the two.*

THEOREM 22. *If a normal congruence be given, not consisting in the circles through two points, the pairs of intersections with any two orthogonal trajectories may be taken as foci of the circles of an associated pseudo-normal congruence.*

The other orthogonal trajectories will then be paired in such a way that the intersections of each circle with a pair of trajectories are mutually inverse in the associated circle.

The pseudo-normal circle congruence enjoys a sort of indestructibility, akin to that of the normal line congruence. Let our congruence be determined by the spheres (y) and (z). We may determine (z') in such a way that

$$(z'\,z') = 1, \qquad (z'\,y) = 0, \qquad (z'\,z) = \cos\theta, \qquad \left(\frac{\partial z'}{\partial u}\,\frac{\partial y}{\partial v}\right) = \left(\frac{\partial z'}{\partial v}\,\frac{\partial y}{\partial u}\right).$$

Here θ is supposed to have a fixed value, the spheres (y) and (z') will determine a second pseudo-normal congruence of such sort that each of its circles is cospherical with one circle of the original pseudo-normal congruence, and makes therewith a fixed angle. The sphere through the circle common to (z) and (z'), making an angle φ with the former is (z''), where

$$z_i'' = \frac{\sin(\theta - \varphi)}{\sin\theta} z_i + \frac{\sin\varphi}{\sin\theta} z_i'.$$

This cuts (y) in a circle coaxal with the two circles already determined thereon, and making an angle φ with the first of them. If, then, φ be constant

$$\left(\frac{\partial z''}{\partial u}\,\frac{\partial y}{\partial v}\right) = \left(\frac{\partial z''}{\partial v}\,\frac{\partial y}{\partial u}\right).$$

THEOREM 23. *If two pseudo-normal congruences be so related that corresponding circles are cospherical and make a fixed angle, then the congruence of circles coaxal with them, and making fixed angles with both, is also pseudo-normal.*

Another theorem of the same sort is obtained as follows. Suppose that we have an analytical complex (three-parameter family) of spheres. A sphere of the complex and its next neighbors are all orthogonal to one sphere which we may speak of as *correlative* to the first. The complex generated by these spheres shall be said to be *correlative* to the first complex. Analytically, if our complex be given parametrically in the form

$$x_i = x_i(u, v, w),$$

the correlative sphere is (y), where

$$\rho y_i = \begin{vmatrix} x_j & \dfrac{\partial x_k}{\partial u} & \dfrac{\partial x_l}{\partial v} & \dfrac{\partial x_m}{\partial w} \end{vmatrix},$$

Since

$$(xy) = \left(y\frac{\partial x}{\partial u}\right) = \left(y\frac{\partial x}{\partial v}\right) = \left(y\frac{\partial x}{\partial w}\right) = \left(x\frac{\partial y}{\partial u}\right) = \left(x\frac{\partial y}{\partial v}\right) = \left(x\frac{\partial y}{\partial w}\right) = 0,$$

we see that (x) and (y) bear a reciprocal relation, or each is the correlative

of the other. This is true in general; there are special cases where (y) may depend on less than three parameters, but these we explicitly exclude by saying that our complex is *non-developable*. Suppose, then, that we have a pseudo-normal congruence. Through each circle will pass at least one sphere of any chosen non-developable complex. We take this for our sphere (y), the correlative sphere is (t), where

$$\rho t_i = \left| y_j \frac{\partial y_k}{\partial u} \frac{\partial y_l}{\partial v} s_m \right|,$$

where s_m is a function of u, v, w. We further put

$$z_i' = \frac{\sin(\varphi - \theta)}{\sin\theta} t_i + \frac{\sin\varphi}{\sin\theta} z_i,$$

so that θ is the angle of (z) and (t), while φ is the angle of (z') and (t), and assume finally that the ratio of $\sin\varphi$ to $\sin\theta$ has the constant value k. We have

$$(yt) = \left(y\frac{\partial t}{\partial u}\right) = \left(y\frac{\partial t}{\partial v}\right) = 0,$$

$$\left(\frac{\partial z'}{\partial u}\frac{\partial y}{\partial v}\right) - \left(\frac{\partial z'}{\partial v}\frac{\partial y}{\partial u}\right) = k\left[\left(\frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) - \left(\frac{\partial z}{\partial v}\frac{\partial y}{\partial u}\right)\right],$$

and this last equation gives the four-dimensional equivalent of the Malus-Dupin theorem.

THEOREM 24. *Through each circle c of a pseudo-normal congruence a sphere is passed belonging to a given non-developable complex in such a way as to generate an analytic congruence of that complex. This sphere meets the correlative sphere in c', and a circle c'' is found coaxal with c and c' so that the sines of the angles of c and c'' with c' bear to one another a constant ratio. The circles c'' will also generate a pseudo-normal congruence.*

We now pass from the non-conformal congruence to the other types noted above. The semi-conformal type has the property that each circle in general position is tangent or paratactic to one adjacent circle. The members of such a congruence are, however, essentially imaginary, and we pass on to the more interesting conformal congruences. Here we have

$$(E - E') : (F - F') : (G - G') = e : \frac{f + f'}{2} : g. \tag{27}$$

Equations (16) become

$$[edu^2 + (f + f')\,du\,dv + gdv^2]\,[\cos 2\varphi + k \sin 2\varphi] = 0.$$

The roots of the second factor give two mutually orthogonal spheres, which we may take for our spheres (y) and (z). A circle cospherical and orthogonal

to a circle in general position and one of its next neighbors must therefore, in general, lie either on (y) or (z). The sphere (y) is thus orthogonal to (z) and to $(z)+(dz)$ so that

$$(ydz) = (zdy) = -\tfrac{1}{2}(dy\,dz) = 0,$$
$$e = f + f' = g = 0. \tag{28}$$

Reverting to (17) we see

$$\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial u}\right) = \left(\frac{\partial\gamma}{\partial u}\frac{\partial\gamma}{\partial u}\right), \quad \left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right) = \left(\frac{\partial\gamma}{\partial u}\frac{\partial\gamma}{\partial v}\right), \quad \left(\frac{\partial\alpha}{\partial v}\frac{\partial\alpha}{\partial v}\right) = \left(\frac{\partial\gamma}{\partial v}\frac{\partial\gamma}{\partial v}\right).$$

This shows that if (α) and (γ) really depend on two parameters, their surfaces are conformally related. There is the additional possibility that all of these terms should vanish identically; (α) and (γ) would then trace two minimal curves. Conversely, let (α) and (γ) trace two conformally related surfaces. We have

$$(d\alpha\,d\alpha) = \sigma\,(d\gamma\,d\gamma).$$

Replacing α_i by $\sqrt[4]{\sigma}\,\alpha_i$ and γ_i by $1/\sqrt[4]{\sigma}\,\gamma_i$

$$(d\alpha\,d\alpha) = (d\gamma\,d\gamma), \qquad (dy\,dz) = 0,$$

we have surely a conformal congruence. We next write

$$y_i' = y_i\cos\varphi + z_i\sin\varphi, \qquad y_i'' = y_i\cos\varphi - z_i\sin\varphi,$$

where φ is constant

$$(dy'\,dy') = (dy''\,dy'').$$

Conversely, if this equation holds, and if φ be a fixed angle

$$(dy\,dz) = 0.$$

We shall say that two congruences of spheres are *conformally related* if the infinitesimal angle of two adjacent spheres of the one is equal to the corresponding infinitesimal angle of the other. We thus get

THEOREM 26. *Through each circle of a conformal congruence we may, in an infinite number of ways, pass two spheres, making a constant angle, and generating conformally related congruences as the circle describes the given congruence.*

THEOREM 25. *If two sphere congruences be conformally related, and corresponding spheres in the two meet at a fixed angle, then the circles of intersection of such spheres will generate a conformal congruence.*

We now make the additional assumption that our circle congruence is focal. The isotropic curves will correspond on the two surfaces generated by the foci. If we take these as our parameter curves u and v we have two conceivable cases

$$(A)\qquad \frac{\partial \gamma_i}{\partial u} = p\alpha_i + q\gamma_i + r\frac{\partial \alpha_i}{\partial v}, \qquad \frac{\partial \gamma_i}{\partial v} = p'\alpha_i + q'\gamma_i + r'\frac{\partial \alpha_i}{\partial u},$$

$$(B)\qquad \frac{\partial \gamma_i}{\partial u} = p\alpha_i + q\gamma_i + r\frac{\partial \alpha_i}{\partial u}, \qquad \frac{\partial \gamma_i}{\partial v} = p'\alpha_i + q'\gamma_i + r'\frac{\partial \alpha_i}{\partial v}.$$

Leaving aside for the moment the question of whether both of these cases actually exist, let us take them up in turn. In case A we have

$$\left(\frac{\partial \alpha}{\partial u}\frac{\partial \gamma}{\partial v}\right) = \left(\frac{\partial \alpha}{\partial v}\frac{\partial \gamma}{\partial u}\right).$$

Our congruence is pseudo-normal, the focal parameters u' and v' will give also the focal directions for the associated normal congruence (by 21), so that they give mutually orthogonal directions of advance for (α) and (γ). We have the partial differential equations, analogous to those previously found for a normal congruence,

$$\frac{\partial \gamma_i}{\partial u'} = bc\alpha_i + b\gamma_i + c\frac{\partial \alpha_i}{\partial u'}, \qquad \frac{\partial \gamma_i}{\partial v'} = b'c'\alpha_i + b'\gamma_i + c'\frac{\partial \alpha_i}{\partial v'}.$$

The relation among the coefficients of α_i, γ_i comes from the equations

$$\left(\alpha\frac{\partial \gamma}{\partial u'}\right) + \left(\gamma\frac{\partial \alpha}{\partial u'}\right) = \left(\alpha\frac{\partial \gamma}{\partial v'}\right) + \left(\gamma\frac{\partial \alpha}{\partial v'}\right) = 0.$$

The condition that our surfaces (α) and (γ) should be conformally related is

$$\left(\frac{\partial \alpha}{\partial u'}\frac{\partial \alpha}{\partial u'}\right)du'^2 + \left(\frac{\partial \alpha}{\partial v'}\frac{\partial \alpha}{\partial v'}\right)dv'^2 = \left(\frac{\partial \gamma}{\partial u'}\frac{\partial \gamma}{\partial u'}\right)du'^2 + \left(\frac{\partial \gamma}{\partial v'}\frac{\partial \gamma}{\partial v'}\right)dv'^2.$$

This gives $c^2 = c'^2$. Now if $c = c'$ we have at once

$$d\gamma_i = P\alpha_i + Q\gamma_i + Rd\alpha_i,$$

each two adjacent circles are cospherical, and we have the circles through two points or on one sphere, which cases we may exclude. Hence we must have

$$c + c' = 0.$$

This shows that the focal spheres, whose coördinates are $c(\alpha) - (\gamma)$, $c'(\alpha) - (\gamma)$ are mutually orthogonal, and the congruence is, by theorem 6, a normal one. Our proeesses here are entirely reversible, and we have the theorems

THEOREM 27. *If a congruence be both conformal and pseudo-normal, it is normal.*

THEOREM 28. *If a congruence be both normal and conformal it is pseudo-normal, or else consists in the circles touching a given circle at a given point.*

With regard to the existence of such congruences, we construct an example as follows. Let a cylinder be given whose elements are all parallel to a given plane, and of such cross section that a sphere whose center moves perpendicularly to the direction of the elements, while it remains tangent to the cylinder and the plane, will trace paths of equal length on the two. The points of contact of the spheres with plane and cylinder will be the required foci. As a matter of fact, however, these congruences are well enough known.* In type B we see that the isotropic parameters are the focal ones. We thus see by theorem B that each circle is tangent to two adjacent circles, or the circles of the congruence are the osculating circles of two different one-parameter families of curves. This much is true, if the congruence exist; unfortunately the present writer has signally failed in all his attempts to answer this interesting question.

§ 4. Complexes of Circles.

Our leading idea in approaching our circle geometry by the Kummer method was to interpret in three dimensions those theorems of the line geometry of four dimensions which are easily reached by an extension of the well-known methods of line geometry. The idea lies close at hand that in changing our line geometry from three to four dimensions, we might have done well to allow ourselves an extra parameter, in other words, to busy ourselves with three-parameter circle systems, or complexes. To such systems we now give our attention. We write

$$y_i = y_i(u_1 u_2 u_3), \qquad z_i = z_i(u_1 u_2 u_3),$$
$$(yy) = (zz) = 1, \qquad (yz) = 0. \tag{29}$$

$$(dy\,dy) - (zdy)^2 \equiv \Sigma a_{ij}\,du_i\,du_j, \qquad a_{ij} = a_{ji},$$
$$(dz\,dz) - (ydz)^2 \equiv \Sigma b_{ij}\,du_i\,du_j, \qquad b_{ij} = b_{ji}, \tag{30}$$
$$(dydz) = \Sigma c_{ij}\,du_i\,du_j.$$

$$\left(\frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_i}\right) - \left(z\frac{\partial y}{\partial u_i}\right)^2 = a_{ii},$$
$$\left(\frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_j}\right) - \left(z\frac{\partial y}{\partial u_i}\right)\left(z\frac{\partial y}{\partial u_j}\right) = a_{ij} + a_{ji} = 2a_{ij} = 2a_{ji} \quad (i \neq j). \tag{31}$$

* The problem of finding a conformal focal congruence is sometimes called the problem of Ribaucour, for it amounts to finding a congruence of spheres which establish a conformal relation between the two nappes of their envelop. For an interesting discussion of the congruences of type A see Darboux, *Sur les surfaces isothermiques*, Annales de l'Ecole Normale, series 3, vol. 16 (1899), pp. 498 ff. Darboux there proves that this is the only type of conformal focal congruence, but his proof is erroneous, as he has acknowledged in a letter to the present writer. He doubted, however, whether any congruences of type B really existed. The theorem that congruences of type A are normal was casually mentioned by Cosserat, *Sur le problème di Ribaucour*, Bulletin de l'Académie des Sciences de Toulouse, vol. 3 (1900), pp. 267 ff.

$$\left(\frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_i}\right) - \left(y\frac{\partial z}{\partial u_i}\right)^2 = b_{ii},$$

(32)

$$\left(\frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_j}\right) - \left(y\frac{\partial z}{\partial u_i}\right)\left(y\frac{\partial z}{\partial u_j}\right) = b_{ij} + b_{ji} = 2b_{ij} = 2b_{ji} \quad (i \neq j).$$

(33)
$$\left(\frac{\partial y}{\partial u_i}\frac{\partial z}{\partial u_j}\right) = c_{ij}.$$

The coefficients are connected by various relations

(34)
$$\sum_{ijkj}\begin{vmatrix}\frac{\partial y_i}{\partial u_1}\frac{\partial y_j}{\partial u_1}\frac{\partial y_k}{\partial u_1}\\ \frac{dy_i}{\partial u_2}\frac{dy_j}{\partial u_2}\frac{\partial y_k}{\partial u_2}\\ \frac{\partial y_i}{\partial u_3}\frac{\partial y_j}{\partial u_3}\frac{\partial y_k}{\partial u_3}\end{vmatrix}\cdot\begin{vmatrix}\frac{\partial z_i}{\partial u_1}\frac{\partial z_j}{\partial u_1}\frac{\partial z_k}{\partial u_1}\\ \frac{\partial z_i}{\partial u_2}\frac{\partial z_j}{\partial u_2}\frac{\partial z_k}{\partial u_2}\\ \frac{\partial z_i}{\partial u_3}\frac{\partial z_j}{\partial u_3}\frac{\partial z_k}{\partial u_3}\end{vmatrix} = \begin{vmatrix}c_{11}\,c_{12}\,c_{13}\\ c_{21}\,c_{22}\,c_{23}\\ c_{31}\,c_{32}\,c_{33}\end{vmatrix},$$

(35)
$$\left|z\,y\frac{\partial y}{\partial u_1}\frac{\partial y}{\partial u_2}\frac{\partial y}{\partial u_3}\right|^2 = |a_{ij}|, \qquad \left|y\,z\frac{\partial z}{\partial u_1}\frac{\partial z}{\partial u_2}\frac{\partial z}{\partial u_3}\right|^2 = |b_{ij}|,$$

$$z_i = \frac{\left|y_j\frac{\partial z_k}{\partial u_1}\frac{\partial z_l}{\partial u_2}\frac{\partial z_m}{\partial u_3}\right|}{\sqrt{|b_{ij}|}}, \qquad y_i = \frac{\left|z_j\frac{\partial y_k}{\partial u_1}\frac{\partial y_l}{\partial u_2}\frac{\partial y_m}{\partial u_3}\right|}{\sqrt{|a_{ij}|}}.$$

If, then, we write

$$B_{kl} \equiv \frac{\partial|b_{ij}|}{\partial b_{kl}}, \qquad A_{kl} \equiv \frac{\partial|a_{ij}|}{\partial a_{kl}},$$

we shall have

(36)
$$a_{ij} = -\frac{\sum_{kl} c_{ik}\,c_{jl}\,B_{kl}}{|b_{ij}|}, \qquad b_{ij} = -\frac{\sum_{kl} c_{ik}\,c_{jl}\,A_{kl}}{|a_{ij}|}.$$

(37)
$$|a_{ij}|\cdot|b_{ij}| = |c_{ij}|^2.$$

We next look for the circles cospherical and orthogonal to a circle in general position, and to one of its next neighbors. Following the method which previously led up to (16) we now have

(38)
$$\sum_{ij} c_{ij}\,du_i\,du_j\,(\cos^2\varphi - \sin^2\varphi) + \sum_{ij}(a_{ij} - b_{ij})\,du_i\,du_j\sin\varphi\cos\varphi = 0.$$

As in the case of the congruence, we must here consider the equations

(39)
$$(d\alpha\,d\alpha) = 0, \qquad (d\gamma\,d\gamma) = 0,$$

where (α) and (γ) are the foci of our circle. When these equations are identical, the complex of circles effect a conformal transformation of space, and the complex is said to be *conformal;* if they have four or fewer common

solutions, the complex is called *non-conformal.* As the equations are irreducible, there is no type of complex corresponding to the semi-conformal congruence. We begin with the non-conformal complex. The circles cospherical and orthogonal to two given circles are indeterminate when, and only when, the given circles are cospherical or are paratactic. The four common solutions of the equations (39) will give four circles adjacent to a given circle and either tangent or paratactic thereto. To determine which of these cases arises, let us see how many circles adjacent to a circle in general position of a complex are cospherical therewith. For cospherical adjacent circles we must have

$$dy_i \cos\varphi + dz_i \sin\varphi \sin\varphi - (y_i \sin\varphi - z_i \cos\varphi)\, d\varphi = 0,$$

$$(ydz) = -(zdy) = d\varphi,$$

$$(40)\quad \sum_{n=1}^{n=3}\left[\frac{\partial y_i}{\partial u_n}\cos\varphi + \frac{\partial z_i}{\partial u_n}\sin\varphi - (y_i \sin\varphi - z_i\cos\varphi)\left(y\frac{\partial z}{\partial u_n}\right)\right]du_n = 0.$$

Multiplying through by $\partial z_i/\partial u_1$, $\partial z_i/\partial u_2$, $\partial z_i/\partial u_3$ and summing, we find

$$(41)\quad \sum_{j=1}^{j=3} c_{ij}\, du_j + \left[\sum_{j=1}^{j=3} b_{ij}\, du_j\right]\tan\varphi = 0 \qquad (i = 1,2,3),$$

$$(42)\quad \begin{vmatrix} (c_{11} + b_{11}\tan\varphi) & \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ \cdots & (c_{33} + b_{33}\tan\varphi) \end{vmatrix} = 0.$$

These equations show that a circle in general position will not, usually, be cospherical with more than three adjacent circles. Conversely, if we have the equations (41), (42) and replace the right-hand side of (40) by c_i while $(ydz) = d\varphi$ we find

$$(yt) = (zt) = \left(\frac{\partial y}{\partial u_1}t\right) = \left(\frac{\partial y}{\partial u_2}t\right) = \left(\frac{\partial y}{\partial u_3}t\right) = 0,$$

hence

$$t_i \equiv 0 \qquad (i = 0 \cdots 4).$$

Lastly, we see that the determination of these three circles depends upon the solution of a (usually) irreducible cubic equation, and so must give different results from the solution of the irreducible quartic arising from (39).

THEOREM 29. *A circle in general position in a non-conformal complex lies usually on three annular surfaces thereof.**

THEOREM 30. *A circle in general position in a non-conformal complex is usually paratactic with four adjacent circles of the complex.*

* This theorem is due to COSSERAT, *Sur le cercle consideré comme élément générateur de l'espace,* Annales de la Faculté des Sciences de Toulouse, vol. 3 (1889) p. E 33.

The qualification *pseudo-normal* may be applied to complexes as well as to congruences of circles. A complex of circles shall be said to be *pseudo-normal* if it be composed of the circles of intersection of corresponding spheres of two correlative complexes. To find the condition that a given complex should be of this type, we put

$$y_i' = y_i \cos\varphi + z_i \sin\varphi, \qquad z_i' = -y_i \sin\varphi + z_i \cos\varphi.$$

The complex will be pseudo-normal if

$$(y'\,dz') = (z'\,dy') = 0.$$

Proceeding exactly as before, we find

$$(ydz) = d\varphi, \tag{43}$$

$$c_{ij} = c_{ji}. \tag{44}$$

We note that if the differential equation for φ have one solution, it will have an infinite number differing by an additive constant.

THEOREM 31. *The circles of a pseudo-normal complex are the intersections of corresponding spheres in a singly infinite system of pairs of correlative complexes. The spheres of any two of these complexes through the same circle make with one another a constant angle.*

We get the necessary and sufficient geometrical conditions for a pseudo-normal complex as follows. Let the three sets of solutions of equations (41), (42) be

$$d^{(1)}u_1\,d^{(1)}u_2\,d^{(1)}u_3\,\varphi_1, \qquad d^{(2)}u_1\,d^{(2)}u_2\,d^{(2)}u_3\,\varphi_2, \qquad d^{(3)}u_1\,d^{(3)}u_2\,d^{(3)}u_3\,\varphi_3.$$

Multiply the three equations (41) through by the corresponding factors $d^{(m)}u_1\,d^{(m)}u_2\,d^{(m)}u_3$ and add; we get equations of the form

$$\sum_{ij} c_{ij}\,d^{(m)}u_i\,d^{(n)}u_j + \tan\varphi_m \sum_{ij} b_{ij}\,d^{(m)}u_i\,d^{(n)}u_j = 0 \qquad (m \neq n),$$

$$\sum_{ij} c_{ij}\,d^{(n)}u_i\,d^{(m)}u_j + \tan\varphi_n \sum_{ij} b_{ij}\,d^{(n)}u_i\,d^{(m)}u_j = 0 \qquad (m \neq n).$$

If now our complex be pseudo-normal, so that (44) are satisfied, we get, by subtraction,

$$\sum_{ij} b_{ij}\,d^{(m)}u_i\,d^{(n)}u_j = 0 \qquad (m \neq n). \tag{45}$$

Conversely, when these equations hold, the complex is pseudo-normal. On the other hand if we mean by the *focal points* of a circle the pairs of points where it meets an adjacent cospherical circle, that is, its pairs of points of contact with the envelop on an annular surface of the complex, we find the

spheres through these pairs of focal points, orthogonal to the given circle exactly as we did for the congruence. We wish to find a linear combination of (y) and (z), which is also a linear combination of $(y)+(dy)$ and $(z)+(dz)$, the differentials being determined by a set of solutions to our equations (41). The three spheres sought will thus be

$$\sum_{k=1}^{k=3}\left[\left(\frac{\partial z}{\partial u_k}y\right)z_i-\frac{\partial z_i}{\partial u_k}\right]du_k^{(1)},\qquad \sum_{k=1}^{k=3}\left[\left(\frac{\partial z}{\partial u_k}y\right)z_i-\frac{\partial z_i}{\partial u_k}\right]du_k^{(2)},$$

$$\sum_{k=1}^{k=3}\left[\left(\frac{\partial z}{\partial u_k}y\right)z_i-\frac{\partial z_i}{\partial u_k}\right]du_k^{(3)}.$$

The equations above give us the necessary and sufficient conditions that these should be mutually orthogonal in pairs.

THEOREM 32. *A necessary and sufficient condition that a complex should be pseudo-normal is that the pairs of focal points on a circle in general position should separate one another harmonically.*

On an arbitrary sphere there will lie a finite number of circles belonging to a given complex, unless all are orthogonal to one sphere. Let two pseudo-normal complexes be determined by the three-parameter sphere systems $(y)\,(z)$ and $(y)\,(z')$ where (z) and (z') make with one another a fixed angle. Then if

$$z_i''=\frac{\sin(\varphi-\theta)}{\sin\theta}z_i+\frac{\sin\varphi}{\sin\theta}z_i',$$

where φ and θ have constant values, we see that (y) and (z'') determine a pseudo-normal complex.

THEOREM 33. *If two pseudo-normal complexes be so related that corresponding circles are cospherical and make a fixed angle with one another, then the complex of circles coaxial respectively with these corresponding pairs, and making fixed angles with them, is also pseudo-normal.*

Suppose next, that (s) is the sphere correlative to the sphere (y); we may write

$$\rho s_i=\left|y_j\ \frac{\partial y_k}{\partial u_1}\ \frac{\partial y_l}{\partial u_2}\ \frac{\partial y_m}{\partial u_3}\right|.$$

If, then, we put

$$z_i'=\frac{\sin(\varphi-\theta)}{\sin\theta}s_i+\frac{\sin\varphi}{\sin\theta}z_i,$$

and require the ratio of $\sin\varphi$ to $\sin\theta$ to be constant, we see that (y) and (z') determine a pseudo-normal complex.

THEOREM 34. *Through each circle of a pseudo-normal complex a sphere is passed belonging to a determinate non-developable complex and the original circle*

is replaced by such a circle coaxial with it and the circle cut by the correlative sphere, that the sines of the angles formed therewith by the original and the replacing circle have a fixed ratio; then will the replacing circles also generate a pseudo-normal complex.

The consideration of the circles adjacent to and cospherical with a circle in general position possesses special features when the complex is conformal, we therefore turn our attention to complexes of that type. Suppose that a pair of foci, which are corresponding points in a conformal transformation of space, are (α) and (γ), and that these are concyclic with the adjacent foci $(\alpha)+(d\alpha)$, $(\gamma)+(d\gamma)$, the circles being then cospherical. If we follow our transformation with an inversion which interchanges (α) and (γ), we have a conformal transformation leaving invariant (α) and the circle through (α), (γ), $(\alpha+d\alpha)$. We are thus led to the consideration of the invariant lineal elements in a conformal transformation of space with a fixed point. We have four possibilities:

1. One proper and two isotropic fixed lineal elements.
2. One proper fixed lineal element, and all elements orthogonal thereto also fixed.
3. All lineal elements of an isotropic plane fixed.
4. All lineal elements fixed.

These four cases for the transformation where (α) is fixed give the possible cases for our given conformal transformation, and a fixed isotropic lineal element through (α) corresponds to the case where (α), (γ), $(\alpha)+(d\alpha)$, $(\gamma)+(d\gamma)$ lie on a null circle, i. e., where adjacent circles are tangent to one another. We thus have

THEOREM 35. *If a circle in general position in a conformal complex be cospherical with but three adjacent circles, it will touch two of them.*

THEOREM 36. *If a circle in general position in a conformal complex be cospherical with three adjacent circles but not tangent to any one of them, it is cospherical with a one-parameter family of adjacent circles, including two of the given ones at least.*

THEOREM 37. *If a circle in general position in a conformal complex be tangent to but one adjacent circle, it will be cospherical with a one-parameter family of adjacent circles including this one, and with no others.*

THEOREM 38. *If a circle in general position in a conformal complex be tangent to three adjacent circles it is cospherical with every adjacent circle, and the complex consists in the totality of circles on a sphere.*

Of course these last four theorems could be restated is more precise form; when we say that adjacent circles are cospherical we mean that our circles can be assembled into the generators of annular surfaces, when we say that they are tangent we mean that we can assemble them into the osculating circles of curves.

Two sphere complexes are said to be *conformally related* when the angle of two adjacent spheres of one is equal to the corresponding infinitesimal angle in the other. We prove exactly as in the case of the conformal congruence:

THEOREM 39. *If a conformal complex be given we may, in an infinite number of ways, pass two spheres through each circle which shall generate conformally related complexes, as the circle generates the given complex.*

THEOREM 40. *If two sphere complexes be conformally related, and corresponding spheres make a fixed angle, their circles of intersection generate a conformal complex.*

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